Carrier Rigidity, Hypercharge, and the Standard Model Packet

from Boundary Polarization

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Abstract

This paper isolates the carrier theorem of TFPT. The carrier polynomial is no longer taken as an entry assumption. Boundary polarization first gives a finite essential two-point carrier $E = E_- \oplus E_+$. The compact determinant/Higgs index fixes $\dim E_+ = 2$, while primitive indecomposable Yukawa type fixes the complementary essential rank $\dim E_- = 3$. The determinant normalized generator is then forced to be

$$Y = -\frac{1}{3}P_- + \frac{1}{2}P_+,$$

and the former carrier equation

$$6Y^2 - Y - \mathbf{1} = 0, \qquad \mathrm{Tr}_E\, Y = 0$$

appears only as its algebraic shadow. From this derived carrier follow the hypercharge vector, the even exterior packet $S^+ = \Lambda^{\mathrm{even}}E$, one chiral Standard-Model family including ν^c, the physical gauge quotient, family counting, admissible occupancy, the compact Higgs index, and the abelian index coefficient.

Scope box: inputs, contribution, non-claims, audit surface

Inputs from previous papers. The primitive boundary kernel $(\tau_{\mathrm{dbl}}, \iota_{\mathrm{C}}, P_{\mathrm{prim}}, [u_\Sigma], c_3)$ from Paper 1.

New theorem contribution. The rigid carrier and packet:

$$E = E_3 \oplus E_2, \quad Y = \mathrm{diag}(-1/3, -1/3, -1/3, 1/2, 1/2), \quad S^+ = \Lambda^{\mathrm{even}}E, \quad G_{\mathrm{phys}} = \frac{SU(3)\times SU(2)\times U(1)_Y}{\mathbb{Z}_6}.$$

Not claimed here. No exact α value, no CKM/PMNS closure, no pole-mass ledger, no OS reconstruction, no cosmology, and no E8 grammar.

Falsification or audit surface. The paper fails if the carrier normal form imports Standard-Model representation data by hand, if the Higgs and branch-Yukawa discharge is not prominent, or if the family and occupancy counts depend on downstream numerics.

Claim contract

Claim. Boundary polarization, compact Higgs selection, and primitive Yukawa type force $\dim E_- = 3$, $\dim E_+ = 2$; the carrier polynomial is a corollary.

Inputs. Primitive kernel from Paper 1 and carrier/exterior-signature discharge.

First assumptions. Essential carrier block, minimal determinant/Higgs index on the positive polarization block, primitive indecomposable Yukawa type; no carrier polynomial assumption.

Proof status. Core theorem plus explicit representation audit.
Kill condition. Higgs/Yukawa rank discharge fails or the determinant-normalized two-point generator is not forced before phenomenological matching.

Contents

1 Carrier Polynomial as a Corollary

The paper does not begin by assuming

$$6Y^2 - Y - \mathbf{1} = 0, \qquad \mathrm{Tr}_E\, Y = 0.$$

It begins with the carrier involution induced by the boundary polarization:

$$\varepsilon_{\mathrm{car}} := \iota_{\mathbb{C}}|_E, \qquad \varepsilon_{\mathrm{car}}^2 = \mathbf{1}, \qquad E = E_- \oplus E_+, \qquad P_\pm = \frac{\mathbf{1} \pm \varepsilon_{\mathrm{car}}}{2}.$$

At this stage there is only a two-point algebra

$$\mathbb{C}[\varepsilon_{\mathrm{car}}] = \{a\mathbf{1} + b\varepsilon_{\mathrm{car}}\}.$$

No $3+2$ split, hypercharge vector, Standard-Model packet, or electromagnetic number has been used.

Theorem 1.1 (Carrier polynomial from boundary polarization)**.** *Let $\mathfrak{T}_{\partial}^{\min}$ be the one-sided admissible boundary datum and let the Calderon polarization induce the finite essential carrier involution above. Assume only:*

(i) the carrier block is essential, so no contractible spectator summand is present on which seam winding, determinant clutching, and primitive local interaction type all vanish;

(ii) unit seam winding selects the minimal nonnegative determinant class on the compactified normal sphere, fixing the positive polarization block by the compact Higgs index;

(iii) the retained branch contains a nonzero primitive indecomposable local trilinear type with two nontrivial fermionic legs and one seam-even bosonic leg.

Then

$$\dim E_+ = 2, \qquad \dim E_- = 3.$$

The determinant-normalized two-point generator is uniquely

$$Y = -\frac{1}{3}P_- + \frac{1}{2}P_+,$$

and hence

$$6Y^2 - Y - \mathbf{1} = 0, \qquad \mathrm{Tr}_E\, Y = 0.$$

Proof spine. Boundary polarization gives only $E = E_- \oplus E_+$. The minimal positive determinant class on the seam-even block gives $L_+ \simeq \mathcal{O}(1)$ on the compactified normal sphere, so $H^0(S^2, \mathcal{O}(1)) \simeq \mathbb{C}^2$ and $\dim E_+ = 2$. The primitive indecomposable Yukawa type is a local type statement, not the later numerical transport realization. With $\dim E_+ = 2$, the seam-even bosonic leg contributes the line $\Lambda^2 E_+$, and closure of the negative factor without a spectator requires

$$E_- \otimes \Lambda^2 E_- \longrightarrow \Lambda^3 E_- = \det E_-,$$

so $\dim E_- = 3$. A determinant-preserving two-point generator has the primitive integer form

$$X = q_- P_- + q_+ P_+, \qquad 3q_- + 2q_+ = 0, \qquad q_- < 0 < q_+, \qquad \gcd(|q_-|, q_+) = 1,$$

whose unique solution is $(q_-, q_+) = (-2, 3)$. Determinant normalization gives $Y = X/6 = -P_-/3 + P_+/2$. The polynomial is the minimal polynomial of these two roots:

$$\left(Y + \frac{1}{3}\right)\left(Y - \frac{1}{2}\right) = 0 \quad \Longleftrightarrow \quad 6Y^2 - Y - \mathbf{1} = 0,$$

and the trace vanishes because $3(-1/3) + 2(1/2) = 0$. □

Remark (General split check)*.* For a general essential split $E_b \oplus E_s$ with determinant-normalized roots $-1/b$ and $1/s$, the two-point generator satisfies

$$bs\, Y^2 + (s - b)Y - \mathbf{1} = 0.$$

The boundary/Higgs/Yukawa rank argument fixes $(b, s) = (3, 2)$, hence the coefficient $6 = 3 \cdot 2$ is not guessed or phenomenologically tuned.

2 Hypercharge and Gauge Quotient

The carrier normal form gives

$$Y = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right),$$

and the internal stabilizer gives the physical quotient

$$G_{\mathrm{phys}} = \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}.$$

The proof must be written as a stabilizer theorem of the carrier datum, not as a reconstruction of a known gauge group.

3 One-Family Packet

The spinor packet is

$$S^+ = \Lambda^{\mathrm{even}} E, \qquad \dim S^+ = 16.$$

This section carries the one-family decomposition including the right-handed neutrino state. It is the place where character identities, trace identities, and the abelian index are stated.

4 Discharging the Rank and Exterior Premises

The editorial risk is that

$$\dim \Lambda^2 E_+ = 1, \qquad \Lambda^2 E_- \cong E_-^\vee$$

look like disguised Standard-Model inputs. For that reason the compact Higgs index and primitive Yukawa type discharge are central rather than appended as a footnote. The value-level transport Yukawa matrices belong downstream; the upstream step here uses only the existence and type of a primitive indecomposable local trilinear.

5 Counting Outputs

The paper records the discrete counting outputs:

$$N_{\mathrm{fam}} = 3, \qquad \Omega_{\mathrm{adm}} = 48, \qquad N_\Phi = 1, \qquad b_1 = \frac{41}{10}.$$

These should be presented as structural consequences of carrier, winding, occupancy, and compact Higgs index closure. They are not yet precision-observable readouts.

6 Main Result

Theorem 6.1 (Carrier rigidity and Standard-Model packet)**.** *Given the primitive boundary kernel of Paper 1, the essential carrier block, the compact Higgs rank selection, and primitive Yukawa type, the admissible carrier branch is the rigid split* $E = E_- \oplus E_+$ *with* $(\dim E_-, \dim E_+) = (3, 2)$*. Renaming these blocks* $E_3 := E_-$ *and* $E_2 := E_+$ *gives the hypercharge vector above. The even exterior packet* $S^+ = \Lambda^{\mathrm{even}} E$ *has dimension* 16*, carries one chiral Standard-Model family including* ν^c*, and has physical stabilizer* G_{phys}*.*

7 Audit Protocol

Audit item	Question
Rank discharge	Do compact Higgs selection and primitive Yukawa type force $(\dim E_-, \dim E_+) = (3, 2)$ before the carrier polynomial is written?
Carrier polynomial	Is $6Y^2 - Y - \mathbf{1} = 0$ presented only as the minimal polynomial of the derived roots?
Hypercharge	Is the normalization fixed before any phenomenological matching?
Packet	Is $S^+ = \Lambda^{\text{even}} E$ derived as the primitive packet, not imported?
Gauge quotient	Is the $\mathbb{Z}_6$ quotient obtained from the stabilizer?
Higgs/Yukawa discharge	Are the exterior-power premises discharged in the body of the paper?

8 Main Technical Development

This section contains the main technical development assigned to this paper by the TFPT 4.5 clean split. Cross-paper background is referenced through dependency and scope boxes; extended backend material is kept in the Technical Companion.

8.1 Carrier algebraic normal form

Lemma 8.1 (Algebraic normal form from exterior signatures)**.** *Let*

$$E = E_- \oplus E_+$$

be a finite chiral carrier with two simple factors of ranks

$$b := \dim E_-, \qquad s := \dim E_+,$$

and let

$$S^+ = \Lambda^{\text{even}} E$$

be the positive half-spinor packet. Let

$$X = q_- P_- + q_+ P_+$$

be a primitive integer two-point generator with

$$q_- < 0 < q_+, \qquad bq_- + sq_+ = 0.$$

Assume only

$$\dim \Lambda^2 E_+ = 1, \qquad \Lambda^2 E_- \cong E_-^\vee.$$

Then necessarily

$$s = 2, \qquad b = 3, \qquad g := b + s = 5.$$

Hence

$$\dim \Lambda^{\text{even}} E = 16.$$

The even exterior algebra decomposes into exactly the six sectors

$$(1,1)_0, \quad (3,2)_{1/6}, \quad (\bar{3},1)_{-2/3}, \quad (\bar{3},1)_{1/3}, \quad (1,2)_{-1/2}, \quad (1,1)_1,$$

and the primitive trace condition therefore has the unique solution

$$(q_-, q_+) = (-2, 3).$$

Consequently

$$X = -2P_3 + 3P_2, \qquad Y := \frac{X}{6} = \operatorname{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right).$$

Proof. From

$$\dim \Lambda^2 E_+ = \binom{s}{2} = 1$$

one gets $s = 2$. For the color factor one always has

$$\Lambda^{b-1} E_- \cong E_-^\vee.$$

The assumption

$$\Lambda^2 E_- \cong E_-^\vee$$

therefore forces $b - 1 = 2$, hence $b = 3$. Thus $g = 5$.

Now

$$S^+ = \Lambda^0 E \oplus \Lambda^2 E \oplus \Lambda^4 E,$$

and for $E = E_3 \oplus E_2$,

$$\Lambda^2 E = \Lambda^2 E_3 \oplus (E_3 \otimes E_2) \oplus \Lambda^2 E_2,$$
$$\Lambda^4 E = (\Lambda^3 E_3 \otimes E_2) \oplus (\Lambda^2 E_3 \otimes \Lambda^2 E_2).$$

These are exactly the six sectors

$$(1,1)_0,\ (3,2)_{1/6},\ (\bar{3},1)_{-2/3},\ (\bar{3},1)_{1/3},\ (1,2)_{-1/2},\ (1,1)_1.$$

Hence

$$\dim S^+ = 2^{5-1} = 16.$$

Finally

$$3q_- + 2q_+ = 0$$

with primitive integers and $q_- < 0 < q_+$ has the unique solution

$$(q_-, q_+) = (-2, 3).$$

□

Corollary 8.2 (One chiral family emerges)**.** *Under the hypotheses of* [TFPT cross-reference: `thm:carrier-minimality-k`], *the positive half-spinor packet S^+ carries exactly one chiral Standard Model family including ν^c.*

Proof. The six-sector decomposition displayed in [TFPT cross-reference: `thm:carrier-minimality-k`] is exactly

$$(1,1)_0 \oplus (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_1,$$

namely one chiral Standard Model family together with ν^c. □

Remark (Logical status of the carrier normal form)*.* The lemma above is the algebraic normal form of the former open carrier target Theorem K. It is a rigidity statement under two exterior signatures:

$$\dim \Lambda^2 E_+ = 1, \qquad \Lambda^2 E_- \cong E_-^\vee.$$

Within those signatures the packet size

$$\dim \Lambda^{\text{even}} E = 16$$

and the one-family decomposition are outputs, while the occupancy identities move behind the carrier closure as derived consistency statements. On the retained branch, the compact Higgs index first fixes the bosonic rank and [TFPT cross-reference: `thm:yukawa-forces-32,cor:yukawa-discharges-k` then discharge these carrier-signature premises internally from the actual branch Yukawa tensor.

Corollary 8.3 (Carrier norm)**.** *On the minimal carrier one has*

$$Y = \operatorname{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right), \qquad \gamma = \operatorname{Tr}_E Y^2 = \frac{5}{6}.$$

Hence from now on

$$E = E_3 \oplus E_2, \qquad X = -2P_3 + 3P_2, \qquad P_3 = \frac{\mathbf{1} - \varepsilon_{\text{car}}}{2}, \qquad P_2 = \frac{\mathbf{1} + \varepsilon_{\text{car}}}{2},$$

and the hypercharge operator is the normalized primitive two-point generator

$$Y = \operatorname{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right) = \frac{X}{6} = -\frac{1}{3}P_3 + \frac{1}{2}P_2 = \frac{1}{12}\mathbf{1} + \frac{5}{12}\varepsilon_{\text{car}}. \tag{1}$$

Lemma 8.4 (Charge-generator / carrier-polarization inversion)**.** *On the rigid carrier, the carrier polarization and the primitive two-point generator are recovered by*

$$\begin{aligned}
\varepsilon_{\text{car}} &= \frac{2X - \mathbf{1}}{5} = \frac{12Y - \mathbf{1}}{5}, &(2)\\
X &= \frac{5\varepsilon_{\text{car}} + \mathbf{1}}{2},\\
(2X - \mathbf{1})^2 &= 25\,\mathbf{1},\\
X^2 - X - 6\,\mathbf{1} &= 0,\\
6Y^2 - Y - \mathbf{1} &= 0. &(3)
\end{aligned}$$

Consequently the carrier projectors are

$$P_2 = \frac{\mathbf{1} + \varepsilon_{\text{car}}}{2} = \frac{2(3Y + \mathbf{1})}{5}, \qquad P_3 = \frac{\mathbf{1} - \varepsilon_{\text{car}}}{2} = \frac{3(\mathbf{1} - 2Y)}{5}.$$

Proof. The affine relations between (X, Y) and ε_{car} are inverted algebraically. Squaring the first relation and using $\varepsilon_{\text{car}}^2 = \mathbf{1}$ gives the quadratic generator polynomial $X^2 - X - 6\,\mathbf{1} = 0$, and dividing by 36 yields the hypercharge polynomial. The projector formulas then follow from $P_2 = (\mathbf{1} + \varepsilon_{\text{car}})/2$ and $P_3 = (\mathbf{1} - \varepsilon_{\text{car}})/2$. □

Lemma 8.5 (Primitive trace-balanced carrier)**.** *Let Y be an endomorphism of a finite-dimensional carrier E such that*

$$6Y^2 - Y - \mathbf{1} = 0, \qquad \operatorname{Tr}_E Y = 0,$$

and assume the realization is primitive in the sense that E is not a nontrivial direct sum of identical trace-balanced Y-modules. Then

$$\operatorname{Spec}(Y) = \left\{-\frac{1}{3}, \frac{1}{2}\right\}, \qquad \dim\ker\left(Y + \frac{1}{3}\mathbf{1}\right) = 3, \qquad \dim\ker\left(Y - \frac{1}{2}\mathbf{1}\right) = 2.$$

Equivalently,

$$\operatorname{rank} E = 5, \qquad E = E_3 \oplus E_2, \qquad Y = -\frac{1}{3}P_3 + \frac{1}{2}P_2.$$

Proof. The quadratic polynomial factors as

$$(2Y - \mathbf{1})(3Y + \mathbf{1}) = 0,$$

so the only eigenvalues are

$$y_+ = \frac{1}{2}, \qquad y_- = -\frac{1}{3}.$$

Write

$$m_+ := \dim\ker(Y - y_+\mathbf{1}), \qquad m_- := \dim\ker(Y - y_-\mathbf{1}).$$

Then

$$0 = \operatorname{Tr}_E Y = \frac{1}{2}\,m_+ - \frac{1}{3}\,m_-,$$

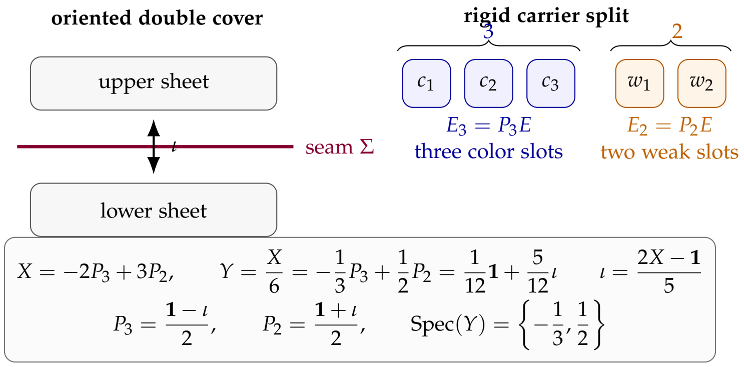


Figure 1. Rigid carrier at a glance. The seam involution, the primitive two-point generator, and the normalized hypercharge operator carry the same hard information: geometrically they separate the two sheets, and algebraically they separate the $3 + 2$ carrier split through the projectors P_3 and P_2.

so

$$3m_+ = 2m_-.$$

Because $\gcd(2,3) = 1$, every positive integer solution has the form

$$(m_+, m_-) = (2k, 3k).$$

Primitivity forces $k = 1$, hence $(m_+, m_-) = (2,3)$. This is exactly the rigid $3 + 2$ split, with the displayed spectral form of Y. □

Remark (Compressed carrier decoder). The carrier can therefore be entered from the shorter algebraic normal form

$$6Y^2 - Y - \mathbf{1} = 0, \qquad \mathrm{Tr}_E\, Y = 0, \qquad \text{primitive realization} \Longrightarrow E = E_3 \oplus E_2.$$

The exterior-signature route of [TFPT cross-reference: `thm:carrier-minimality-k`] remains the theorem-level discharge of these algebraic hypotheses from the actual branch geometry.

The hard carrier package is summarized visually in Figure 1. The point is that the same minimal data are readable in four equivalent ways: geometrically as an oriented double cover with deck reflection and boundary polarization, algebraically as the trace-balanced quadratic carrier decoder, combinatorially as the rigid $3 + 2$ split, and spectrally as the primitive two-point generator X with normalized hypercharge $Y = X/6$.

Remark (Two-point carrier algebra). Because the minimal carrier satisfies $6Y^2 - Y - \mathbf{1} = 0$, every analytic function of Y reduces to

$$f(Y) = A_f\,\mathbf{1} + B_f\,Y, \qquad A_f = \frac{2f(1/2) + 3f(-1/3)}{5}, \qquad B_f = \frac{6\big(f(1/2) - f(-1/3)\big)}{5}.$$

Thus the hard carrier is effectively a two-point algebra supported on the spectral values $-1/3$ and $1/2$, and the projectors P_3, P_2 are already polynomial functions of Y.

Remark (Factorized carrier polynomial and carrier discriminant). For a general split carrier $E = E_b \oplus E_s$ with total rank $g := b + s$ and eigenvalues $-1/b$ and $1/s$, the carrier generator satisfies

$$\left(Y + \frac{1}{b}\right)\left(Y - \frac{1}{s}\right) = 0 \quad \Longleftrightarrow \quad bs\,Y^2 + (s-b)Y - \mathbf{1} = 0.$$

Its discriminant is

$$\Delta_Y = (s-b)^2 + 4bs = (b+s)^2 = g^2.$$

Thus the square root of the carrier discriminant is the carrier rank itself. Moreover,

$$Y = \frac{b-s}{2bs}\,\mathbf{1} + \frac{g}{2bs}\,\varepsilon_{\text{car}}, \qquad \varepsilon_{\text{car}} = \frac{2bs\,Y - (b-s)\,\mathbf{1}}{g}.$$

On the minimal branch $(b,s) = (3,2)$, this reduces to

$$6Y^2 - Y - \mathbf{1} = (2Y - \mathbf{1})(3Y + \mathbf{1}), \qquad \Delta_Y = 25 = 5^2, \qquad \varepsilon_{\text{car}} = \frac{12Y - \mathbf{1}}{5}.$$

Hence the denominator 5 in the hypercharge–seam inversion is exactly $\sqrt{\Delta_Y}$, not an additional inserted constant.

This does not replace the stabilizer proof of

$$G_{\text{car}} = S(U(3) \times U(2)),$$

which is the actual source of the internal carrier stabilizer theorem in the present manuscript. What the factorization and discriminant do show exactly is that the hard carrier already supports the distinguished spectral values

$$Y = \frac{1}{2}, \qquad Y = -\frac{1}{3},$$

and that the inversion denominator is fixed algebraically by the carrier discriminant.

The two carrier invariants used repeatedly below are

$$\gamma = \operatorname{Tr}_E Y^2 = \frac{5}{6}, \qquad B = \frac{\operatorname{rank} E_3}{\operatorname{rank} E_2} = \frac{3}{2}.$$

Corollary 8.6 (Carrier norm from the carrier polynomial)**.** *The carrier polynomial already fixes*

$$\gamma = \operatorname{Tr}_E Y^2 = \frac{1}{6}\operatorname{Tr}_E(Y + \mathbf{1}) = \frac{5}{6},$$

since $\operatorname{Tr}_E Y = 0$ *and* $\operatorname{rank} E = 5$*.*

Corollary 8.7 (Minimal carrier compression)**.** *On the minimal carrier branch,*

$$B\gamma = \frac{3}{2} \cdot \frac{5}{6} = \frac{5}{4}.$$

We package the hard kernel as

$$\mathcal{C}_{\min} = (E_3 \oplus E_2,\ X,\ Y,\ S^+,\ \gamma,\ B).$$

8.2 Internal carrier stabilizer and one-family packet

Theorem 8.8 (Internal carrier stabilizer)**.** *The determinant-preserving connected stabilizer of the minimal carrier split is*

$$G_{\text{car}} = S(U(3) \times U(2)) \cong \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}.$$

The corresponding Lie algebra factorization

$$\mathfrak{g}_{\text{SM}} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$$

is only the differential, local form of the same statement; throughout the manuscript this quotient is intended as the internal carrier-stabilizer form whenever a group symbol is written, and the local algebra notation is used only when explicitly stated.

Proof. An automorphism preserving the split $E_3 \oplus E_2$ acts by a pair $(u_3, u_2) \in U(3) \times U(2)$. Requiring determinant preservation on the full carrier gives

$$\det(u_3)\det(u_2) = 1,$$

so the connected stabilizer is exactly $S(U(3) \times U(2))$.

Define

$$\Phi : SU(3) \times SU(2) \times U(1)_Y \longrightarrow S(U(3) \times U(2)), \qquad \Phi(g_3, g_2, z) = \big(z^2 g_3,\ z^{-3} g_2\big).$$

This is a well-defined group homomorphism because the exponents are integers, and

$$\det(z^2 g_3)\det(z^{-3} g_2) = z^6 z^{-6} = 1.$$

To prove surjectivity, take $(u_3, u_2) \in S(U(3) \times U(2))$. Choose $z \in U(1)$ with

$$z^6 = \det(u_3) = \det(u_2)^{-1}.$$

Then

$$g_3 := z^{-2} u_3 \in SU(3), \qquad g_2 := z^3 u_2 \in SU(2),$$

and therefore

$$\Phi(g_3, g_2, z) = (u_3, u_2).$$

If $\Phi(g_3, g_2, z) = (\mathbf{1}_3, \mathbf{1}_2)$, then

$$g_3 = z^{-2}\mathbf{1}_3, \qquad g_2 = z^3 \mathbf{1}_2.$$

The conditions $g_3 \in SU(3)$ and $g_2 \in SU(2)$ imply $z^6 = 1$. Hence

$$\ker\Phi = \big\{\big(z^{-2}\mathbf{1}_3,\ z^3\mathbf{1}_2,\ z\big) : z^6 = 1\big\} \cong \mathbb{Z}_6.$$

Therefore

$$S(U(3) \times U(2)) \cong \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}.$$

□

Theorem 8.9 (Faithful physical global gauge group). *The action of*

$$S(U(3) \times U(2))$$

on the admissible matter-plus-Higgs package

$$\mathcal{R}_{\text{adm}} := S^+ \oplus E_2$$

is faithful. Therefore

$$G_{\text{phys}} = G_{\text{car}} = S(U(3) \times U(2)).$$

Equivalently,

$$G_{\text{phys}} \cong \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}.$$

No further nontrivial quotient survives on the physical matter-plus-Higgs package.

Proof. Let $(u_3, u_2) \in S(U(3) \times U(2))$ act trivially on $\mathcal{R}_{\mathrm{adm}}$. The Higgs sector is E_2, so triviality there implies

$$u_2 = \mathbf{1}_2.$$

The matter packet contains the quark doublet sector

$$Q = E_3 \otimes E_2.$$

Since $u_2 = \mathbf{1}_2$, trivial action on Q forces

$$u_3 = \mathbf{1}_3.$$

Hence the kernel is trivial. The quotient from $SU(3) \times SU(2) \times U(1)$ to $S(U(3) \times U(2))$ already has kernel $\mathbb{Z}_6$ by [TFPT cross-reference: `thm:global-sm-theorem`], so this is the exact physical global form. □

Throughout the manuscript, $S(U(3) \times U(2))$ is therefore read not only as the internal carrier stabilizer but as the physical global gauge group selected by the faithful action on the admissible matter-plus-Higgs package.

Matter is read from the positive half-spinor

$$S^+ = \Lambda^{\mathrm{even}} E, \qquad \dim S^+ = 16.$$

Theorem 8.10 (One carrier family from the exterior packet)**.** *The positive half-spinor of the minimal carrier decomposes as*

$$S^+ \cong (1,1)_0 \oplus (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_1,$$

that is, exactly one chiral Standard Model family with ν^c. Equivalently, in the familiar unified packaging language,

$$S^+ \cong \mathbf{1} \oplus \mathbf{10} \oplus \bar{\mathbf{5}}.$$

Proof. One has

$$S^+ = \Lambda^0 E \oplus \Lambda^2 E \oplus \Lambda^4 E.$$

The $\Lambda^0 E$ sector is immediately

$$\Lambda^0 E \cong (1,1)_0,$$

which is identified with ν^c in left-chiral notation. For the minimal carrier,

$$\Lambda^2 E \cong \Lambda^2 E_3 \oplus (E_3 \otimes E_2) \oplus \Lambda^2 E_2.$$

Here

$$\Lambda^2 E_3 \cong \bar{3}, \qquad E_3 \otimes E_2 \cong (3,2), \qquad \Lambda^2 E_2 \cong 1,$$

and the hypercharges are obtained by adding the carrier charges from

$$Y = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right).$$

Therefore

$$\Lambda^2 E \cong (\bar{3},1)_{-2/3} \oplus (3,2)_{1/6} \oplus (1,1)_1.$$

Because the full determinant is trivial in $S(U(3) \times U(2))$, one has $\Lambda^5 E \cong 1$ and hence

$$\Lambda^4 E \cong E^*.$$

Exterior component	SM representation	Hypercharge	Field content	Mult.
$\Lambda^0 E$	$(1,1)_0$	0	ν^c	1
$\Lambda^2 E_3$	$(\bar{3},1)_{-2/3}$	$-2/3$	u^c	3
$E_3 \otimes E_2$	$(3,2)_{1/6}$	$1/6$	Q_L	6
$\Lambda^2 E_2$	$(1,1)_1$	1	e^c	1
$\Lambda^3 E_3$ $\otimes E_2$	$(1,2)_{-1/2}$	$-1/2$	L_L	2
$\Lambda^2 E_3$ $\otimes \Lambda^2 E_2$	$(\bar{3},1)_{1/3}$	$1/3$	d^c	3
			Total multiplicity	16

Table 1. One chiral family from $S^+ = \Lambda^{\text{even}} E$, written in left-chiral notation with charge-conjugate singlets.

Equivalently,

$$\Lambda^4 E \cong (\Lambda^3 E_3 \otimes E_2) \oplus (\Lambda^2 E_3 \otimes \Lambda^2 E_2).$$

Using $\Lambda^3 E_3 \cong 1$, $\Lambda^2 E_2 \cong 1$, and $\Lambda^2 E_3 \cong E_3^* \cong \bar{3}$, one gets

$$\Lambda^4 E \cong (1,2)_{-1/2} \oplus (\bar{3},1)_{1/3}.$$

Collecting the three even-degree sectors yields

$$S^+ \cong (1,1)_0 \oplus (\bar{3},1)_{-2/3} \oplus (3,2)_{1/6} \oplus (1,1)_1 \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2},$$

with total multiplicity $1+3+6+1+3+2 = 16$. The $\Lambda^2 E$ sector is the standard left-chiral **10** of $SU(5)$, the $\Lambda^4 E$ sector is the standard left-chiral $\bar{\mathbf{5}}$, and the $\Lambda^0 E$ sector is the singlet **1**. □

In left-chiral notation the one-family decomposition takes the explicit form shown in [TFPT cross-reference: `tab:one-family`]. This is the place where the manuscript must be explicit: the carrier theorem is meaningful only if the reader can see how the sixteen states are organized.

Proposition 8.11 (Exact parity-code realization of the family packet)**.** Identifying each wedge monomial in ΛE with its occupation word

$$x = (x_1, \ldots, x_5) \in \{0,1\}^5,$$

the positive half-spinor is exactly the even-parity subspace

$$S^+ \cong \operatorname{span}\Big\{ |x\rangle : \sum_{i=1}^{5} x_i \equiv 0 \pmod 2 \Big\}.$$

Equivalently,

$$S^+ = \big(\Lambda^{\text{even}} E_3 \otimes \Lambda^{\text{even}} E_2\big) \oplus \big(\Lambda^{\text{odd}} E_3 \otimes \Lambda^{\text{odd}} E_2\big),$$

and each summand has dimension 8.

Proof. Choose a basis of E adapted to the split $E_3 \oplus E_2$. Exterior monomials are then in bijection with occupation words $x \in \{0,1\}^5$, and the exterior degree is exactly the Hamming weight $\sum_i x_i$. Hence $\Lambda^{\text{even}} E$ is the span of the even-parity words. Splitting the degree into the E_3 and E_2 contributions gives the direct-sum decomposition above. Since

$$\dim \Lambda^{\text{even}} E_3 = 1 + \binom{3}{2} = 4, \qquad \dim \Lambda^{\text{odd}} E_3 = \binom{3}{1} + \binom{3}{3} = 4,$$

and

$$\dim \Lambda^{\text{even}} E_2 = 1 + \binom{2}{2} = 2, \qquad \dim \Lambda^{\text{odd}} E_2 = \binom{2}{1} = 2,$$

each tensor-product sector has dimension 8. □

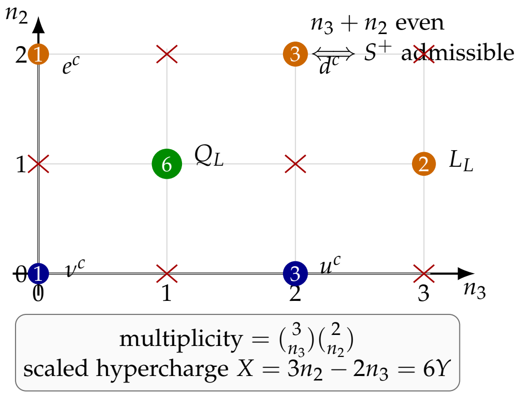


Figure 2. Occupancy map of the family packet. The admissible sectors are exactly the even lattice points in (n_3, n_2), with multiplicities $(1, 3, 6, 1, 2, 3)$ and field labels matching [TFPT cross-reference: `tab:one-family`]. This is the visual meeting point of parity lock, split weight enumerator, and family hypercharge arithmetic.

Remark (Exact parity lock, not a loose code analogy). The family packet is therefore not merely code-like: at the level of occupation kinematics it is exactly the classical even-parity $[5, 4, 2]$ code on five bits. What the present paper still does *not* claim is a dynamical error-correction theorem for realistic noise channels. The exact statement is the parity lock itself. In the maximally mixed state on the 16 occupation words, eight states lie in the $(\text{even}, \text{even})$ sector and eight in (odd, odd), so the coarse parity observables of the 3-side and 2-side carry exactly one bit of mutual information.

The family packet can also be read as a two-dimensional occupancy map, shown in Figure 2. This is where the one-family table, the even-parity code, the split weight enumerator, and the $X = 6Y$ arithmetic all meet in one glance.

8.3 Character, trace identities, and the abelian index

Proposition 8.12 (Split weight enumerator and one-family character)**.** For the present paragraph let

$$\chi_{S^+}(t) := \mathrm{Tr}_{S^+}(e^{tY}).$$

Define also the split weight enumerator

$$W(u, v) := \sum_{\substack{0 \le n_3 \le 3 \\ 0 \le n_2 \le 2 \\ n_3 + n_2 \text{ even}}} \binom{3}{n_3}\binom{2}{n_2} u^{n_3} v^{n_2}.$$

Then

$$W(u, v) = 1 + 3u^2 + 6uv + v^2 + 2u^3 v + 3u^2 v^2 = \frac{1}{2}\Big[(1+u)^3(1+v)^2 + (1-u)^3(1-v)^2\Big]. \quad (4)$$

Moreover,

$$\chi_{S^+}(t) = \frac{1}{2}\Big[\big(1+e^{-t/3}\big)^3\big(1+e^{t/2}\big)^2 + \big(1-e^{-t/3}\big)^3\big(1-e^{t/2}\big)^2\Big] = W(e^{-t/3}, e^{t/2}). \quad (5)$$

Consequently,

$$\chi_{S^+}(0) = 16, \qquad \chi'_{S^+}(0) = \mathrm{Tr}_{S^+} Y = 0, \qquad \chi''_{S^+}(0) = \mathrm{Tr}_{S^+} Y^2 = \frac{10}{3}, \qquad \chi'''_{S^+}(0) = \mathrm{Tr}_{S^+} Y^3 = 0.$$

Remark (Master generating function for the one-family packet). The even-parity split weight enumerator

$$W(u, v) = \frac{1}{2}[(1+u)^3(1+v)^2 + (1-u)^3(1-v)^2]$$

is the master generator of the family packet. The one-family character is obtained by the specialization $u = e^{-t/3}$ and $v = e^{t/2}$, while all trace moments of Y are recovered by differentiating at $t = 0$.

Proof. The pairs (n_3, n_2) allowed by even total degree are

$$(0,0),\ (2,0),\ (1,1),\ (0,2),\ (3,1),\ (2,2),$$

with multiplicities

$$1,\ 3,\ 6,\ 1,\ 2,\ 3.$$

This gives the explicit polynomial formula for $W(u,v)$. Equivalently, the parity projector

$$\frac{1}{2}(1 + (-1)^{n_3+n_2})$$

acting on $(1+u)^3(1+v)^2$ yields Equation (4). Substituting

$$u = e^{-t/3}, \qquad v = e^{t/2}$$

gives Equation (5). For an even exterior algebra one also has

$$\mathrm{Tr}_{\Lambda^{\mathrm{even}}E}(e^{tY}) = \frac{1}{2}\Big[\prod_{j=1}^{5}(1+e^{tq_j}) + \prod_{j=1}^{5}(1-e^{tq_j})\Big],$$

with carrier charges $q_j \in \{-1/3, -1/3, -1/3, 1/2, 1/2\}$. Differentiating at $t = 0$ yields the stated dimension and trace identities. □

Corollary 8.13 (Multiplicative character as six-sector decoder). *Let*

$$X := 6Y.$$

Then the multiplicative one-family character

$$\Xi_{S^+}(t) := \mathrm{Tr}_{S^+}(t^X)$$

is

$$\Xi_{S^+}(t) = \frac{1}{2}\Big[(1+t^{-2})^3(1+t^3)^2 + (1-t^{-2})^3(1-t^3)^2\Big] = t^6 + 3t^2 + 6t + 1 + 2t^{-3} + 3t^{-4}. \quad (6)$$

Consequently

$$\Xi_{S^+}(1) = 16, \qquad (t\partial_t)\Xi_{S^+}(1) = \mathrm{Tr}_{S^+} X = 0, \qquad (t\partial_t)^2\Xi_{S^+}(1) = \mathrm{Tr}_{S^+} X^2 = 120 = 5!.$$

Thus the six sectors and their multiplicities are read off directly from the monomials of Equation (6).

Proof. Since $X = 3N_2 - 2N_3$, every admissible occupancy pair contributes the monomial $t^{3n_2-2n_3}$. Therefore

$$\Xi_{S^+}(t) = W(t^{-2}, t^3),$$

and Equation (4) gives the displayed factorized form. Expanding yields the six monomials shown above. Applying $t\partial_t$ once and twice at $t = 1$ recovers the first two trace moments of X. □

Remark (Table as readout, character as proof device). The one-family table remains useful as a reader-facing decoding sheet, but the whole six-sector arithmetic is already compressed into the single multiplicative character Equation (6).

Remark (Six admissible occupancy sectors). The six monomials in Equation (4) are exactly the six allowed occupancy pairs (n_3, n_2) of the family packet. This is the shortest combinatorial route from the split carrier to the six Standard Model multiplet sectors listed in [TFPT cross-reference: `tab:one-family`].

The carrier packet therefore reproduces the standard trace cancellations

$$\mathrm{Tr}_{S^+} Y = 0, \qquad \mathrm{Tr}_{S^+} Y^3 = 0,$$

and packages them together with the quadratic trace in one formula. Since $\mathrm{Tr}_{S^+} Y^2 = 4\gamma = 10/3$ and $\mathrm{Tr}_\Phi Y^2 = 1/2$ for one weak doublet with hypercharge $1/2$, the hypercharge trace coefficient organizes as

$$b_1(N_{\mathrm{fam}}, N_\Phi) = \frac{2}{5} N_{\mathrm{fam}} \,\mathrm{Tr}_{S^+} Y^2 + \frac{1}{5} N_\Phi \,\mathrm{Tr}_\Phi Y^2 = \frac{8}{5} N_{\mathrm{fam}} \gamma + \frac{1}{10} N_\Phi = \frac{4}{3} N_{\mathrm{fam}} + \frac{1}{10} N_\Phi. \tag{7}$$

On the canonical branch $(N_{\mathrm{fam}}, N_\Phi) = (3, 1)$ this gives

$$b_1 = \frac{41}{10}.$$

The abelian index is therefore not inserted as an isolated Standard Model datum; it is a carrier-level output once the family and Higgs counts are declared.

Remark (Canonical-branch Diophantine lock). If one imposes $b_1 = 41/10$ on nonnegative integer data, then

$$40 N_{\mathrm{fam}} + 3 N_\Phi = 123.$$

The physically nontrivial solution is $(N_{\mathrm{fam}}, N_\Phi) = (3, 1)$, so the canonical branch is already sharply constrained at the level of the trace arithmetic.

Proposition 8.14 (Family hypercharge arithmetic and compression)**.** Let N_3 and N_2 be the occupation-number operators on the E_3 and E_2 slots, restricted to S^+, and define the scaled hypercharge operator

$$X := 6Y = 3N_2 - 2N_3.$$

Then on S^+ one has

$$X \equiv N_2 \equiv N_3 \pmod 2, \tag{8}$$
$$X \equiv N_3 \pmod 3, \tag{9}$$
$$X \equiv 3(N_2 + N_3) \pmod 5. \tag{10}$$

In particular, the even exterior degree $k = N_2 + N_3 \in \{0, 2, 4\}$ is read off from the residue class of X modulo 5: if $X \equiv r \pmod 5$ with $r \in \{0, 1, 2\}$, then $k = 2r$, so

$$X \bmod 5 = 0, 1, 2 \qquad \Longleftrightarrow \qquad \Lambda^0 E,\ \Lambda^2 E,\ \Lambda^4 E.$$

Moreover,

$$\mathrm{Spec}(X|_{S^+}) = \{0, -4, 1, 6, -3, 2\},$$

and therefore

$$X(X+4)(X-1)(X-6)(X+3)(X-2) = 0 \qquad \text{on } S^+. \tag{11}$$

Proof. Each occupied E_3 mode contributes hypercharge $-1/3$ and each occupied E_2 mode contributes $+1/2$, so multiplying by 6 gives $X = 3N_2 - 2N_3$. Modulo 2, the term $-2N_3$ vanishes, so $X \equiv N_2$. Because S^+ has even total degree, $N_2 \equiv N_3 \pmod 2$, which proves Equation (8). Modulo 3, one has $3N_2 \equiv 0$ and $-2 \equiv 1$, giving Equation (9). Modulo 5, the congruence $-2 \equiv 3$ yields Equation (10). Since on S^+ the even degree is $k = N_2 + N_3 \in \{0, 2, 4\}$, the possible residues are $3k \equiv 0, 1, 2 \pmod 5$, from which $k = 2r$ with $r = X \bmod 5$ follows. Evaluating $X = 3n_2 - 2n_3$ on the six admissible occupancy pairs (n_3, n_2) gives the listed spectrum. The polynomial Equation (11) is then the spectral polynomial with these six simple roots. □

Corollary 8.15 (Five-fold phase filters for even degree)**.** *Let*

$$\zeta_5 = e^{2\pi i/5}, \qquad U_5 := e^{2\pi i X/5}.$$

Then the even-degree projectors are the discrete Fourier filters

$$P_{\Lambda^{2m}E} = \frac{1}{5}\sum_{r=0}^{4} \zeta_5^{-mr} U_5^r, \qquad m = 0, 1, 2.$$

Proof. By [TFPT cross-reference: `prop:family-hypercharge-compression`], the eigenspaces $\Lambda^{2m}E$ carry the residue class $X \equiv m \pmod 5$. Hence U_5 has eigenvalue ζ_5^m on $\Lambda^{2m}E$, and the standard Fourier projector formula on the cyclic group $\mathbb{Z}_5$ gives the claim. □

Remark (From the two-point carrier to the six-point family algebra)*.* The minimal carrier itself is a two-point algebra, but the family packet lifts the scaled hypercharge to a six-point algebra. Therefore every analytic function of X on S^+ reduces to a polynomial of degree at most five. If

$$M_n := \mathrm{Tr}_{S^+}(X^n),$$

then Equation (11) implies the finite recursion

$$M_{n+6} = 2M_{n+5} + 31M_{n+4} - 20M_{n+3} - 156M_{n+2} + 144M_{n+1}.$$

In this sense the whole hypercharge statistics of one family is finitely compressed into six spectral values.

9 Compression identities and numerical comparison of alternative discrete worlds

This section collects the algebraic compressions that turn the carrier packet and kernel numerics from a list of internally consistent outputs into an overdetermined structure. These compression identities are derived consistency statements on the retained branch: their formal derivation closes only once the bosonic-rank and branch-Yukawa carrier theorems have fixed the rigid $3 + 2$ split. By contrast, the discrete-world table below is a numerical comparison layer: it is not part of the rigid theorem surface and should be read as an appendix-style pressure test on the carrier packet.

9.1 Derived $48 = 4 \cdot 12$ identity and the 2-3-5 arithmetic

Proposition 9.1 (Derived $48 = 4 \cdot 12$ identity on the retained branch)**.** On the retained branch, once the family multiplicity and carrier ranks have been fixed, one recovers

$$\Omega_{\mathrm{adm}} = N_{\mathrm{fam}} \dim S^+ = (g-1)\dim \mathfrak{g}_{\mathrm{car}} = (g-1)(b^2 + s^2 - 1)$$

with

$$N_{\mathrm{fam}} = 3, \qquad \dim S^+ = 16, \qquad (b, s) = (3, 2), \qquad g = 5.$$

Equivalently, the former rank-lock formula is a derived consistency identity evaluated at the closed branch point:

$$3 \cdot 2^{g-1} = (g-1)(b^2 + s^2 - 1). \tag{12}$$

Proof. Theorem 10.3 gives the physical family multiplicity

$$N_{\text{fam}} = \dim \mathcal{H}_F = 3.$$

The carrier closure is completed later by [TFPT cross-reference: `cor:bosonic-rank-two,thm:yukawa-forces-3` which fix

$$(b,s) = (3,2), \qquad g = 5, \qquad \dim S^+ = 16.$$

Therefore

$$\Omega_{\text{adm}} = N_{\text{fam}} \dim S^+ = 3 \cdot 16 = 48.$$

For the same minimal branch,

$$\dim \mathfrak{g}_{\text{car}} = \dim \mathfrak{su}(3) + \dim \mathfrak{su}(2) + \dim \mathfrak{u}(1) = 12.$$

Therefore

$$\Omega_{\text{adm}} = 48 = (g-1)\dim \mathfrak{g}_{\text{car}} = 4 \cdot 12,$$

which is exactly the displayed derived identity written in branch variables. The former occupancy balance is thus a consistency check rather than an input assumption. □

Remark (The 2-3-5 arithmetic). The hard carrier packet rests on exactly three discrete numbers: 2 (the double cover), 3 (the color rank and family count), and 5 (the carrier rank). From these three atoms one recovers

$$\gamma = \frac{5}{6} = \frac{5}{2\cdot 3}, \qquad B = \frac{3}{2}, \qquad B\gamma = \frac{5}{4} = \frac{5}{2^2},$$

$$\Omega_{\text{adm}} = 3 \cdot 2^4 = 48, \qquad D_{\text{start}} = 5 \cdot 12 = 60, \qquad \frac{D_{\text{start}}}{\Omega_{\text{adm}}} = \frac{5}{4} = B\gamma,$$

and the defect coefficients $\delta_{\text{top}} = 48\, c_3^4$ and $\delta_2 = \frac{5}{4}\,\delta_{\text{top}}^2 = 2880\, c_3^8$ with $2880 = 60 \cdot 48$. The persistent appearance of $5/4$ as carrier asymmetry $\times$ norm, second defect stage, and ladder-to-occupancy ratio is therefore not three coincidences but one rational shadow of the 2-3-5 triad.

Lemma 9.2 (Universal 5/4 compression quotient on the minimal branch)**.** *Define*

$$\rho_{5/4} := B\gamma.$$

On the canonical minimal branch,

$$\rho_{5/4} = B\gamma = \frac{\delta_2}{\delta_{\text{top}}^2} = \frac{D_{\text{start}}}{\Omega_{\text{adm}}} = \frac{g_{\text{car}}}{g_{\text{car}} - 1} = \frac{5}{4}.$$

Thus $5/4$ *is the universal compression quotient of the minimal branch rather than a recurring numerical motif.*

Proof. The identity $B\gamma = \frac{5}{4}$ is Corollary 8.7. The defect relation gives

$$\frac{\delta_2}{\delta_{\text{top}}^2} = \frac{5}{4}.$$

On the canonical branch,

$$\frac{D_{\text{start}}}{\Omega_{\text{adm}}} = \frac{60}{48} = \frac{5}{4},$$

and with $g_{\text{car}} = 5$ one also has

$$\frac{g_{\text{car}}}{g_{\text{car}} - 1} = \frac{5}{4}.$$

Hence all displayed ratios agree on the same exact value. □

Remark (Interpretive status of the quotient). The quotient itself is exact. Any holographic or spacetime-dimensional reading of the denominator $g_{\rm car} - 1 = 4$ remains interpretive in the present manuscript rather than theorem-level.

Corollary 9.3 (Rank-5 master compression on the retained branch)**.** *On the retained branch one has* $g_{\rm car} = 5$. *Consequently*

$$N_{\rm fam} = \frac{g_{\rm car}+1}{2} = 3, \qquad \Omega_{\rm adm} = (g_{\rm car}+1)\, 2^{g_{\rm car}-2} = 48,$$

$$\Omega_{\rm adm}\gamma = g_{\rm car}\, 2^{g_{\rm car}-2} = 40, \qquad b_1 = \frac{g_{\rm car}\, 2^{g_{\rm car}-2}+1}{10} = \frac{41}{10}, \qquad 10\, b_1 = 41.$$

Thus the closed family count, admissible occupancy, weighted occupancy, and abelian index all compress to the single carrier rank.

Proof. By [TFPT cross-reference: `cor:bosonic-rank-two,thm:yukawa-forces-32`], the retained carrier has

$$g_{\rm car} = 5.$$

By Theorem 10.3, the family block fixes $N_{\rm fam} = 3$, hence

$$\Omega_{\rm adm} = N_{\rm fam}\, 2^{g_{\rm car}-1} = 3 \cdot 2^4 = 48.$$

The carrier norm corollary gives $\gamma = 5/6$, so

$$\Omega_{\rm adm}\gamma = 48 \cdot \frac{5}{6} = 40.$$

Finally the closed Higgs branch has $N_\Phi = 1$, and [TFPT equation: `eq:41-chain-main`] yields

$$10\, b_1 = \Omega_{\rm adm}\gamma + N_\Phi = 40 + 1 = 41.$$

□

Corollary 9.4 (Pythagorean compression of the abelian index on the retained branch)**.** *On the retained branch,*

$$\mathcal{I}_{41} = 10\, b_1 = 2g_{\rm car}(g_{\rm car}-1) + 1 = g_{\rm car}^2 + (g_{\rm car}-1)^2.$$

In particular, since $g_{\rm car} = 5$ *on the retained branch, one gets*

$$\mathcal{I}_{41} = 5^2 + 4^2 = 41.$$

Proof. By [TFPT cross-reference: `cor:rank-5-master-compression`], one has $g_{\rm car} = 5$ and

$$10\, b_1 = g_{\rm car}\, 2^{g_{\rm car}-2} + 1 = 5 \cdot 2^3 + 1 = 41.$$

$$2g_{\rm car}(g_{\rm car}-1) + 1 = 2 \cdot 5 \cdot 4 + 1 = 41.$$

Since on the retained branch

$$2g_{\rm car}(g_{\rm car}-1) + 1 = g_{\rm car}^2 + (g_{\rm car}-1)^2,$$

the claim follows. □

Remark (The number 4 as a cross-cutting echo). In addition to the 2-3-5 triad, the integer 4 appears at four structurally distinct levels:

$$F_{\rm patch} = 4, \qquad g_{\rm car} - 1 = 4, \qquad \frac{\Omega_{\rm adm}}{\dim \mathfrak{g}_{\rm SM}} = \frac{48}{12} = 4, \qquad \frac{|R(E_8)|}{D_{\rm start}} = \frac{240}{60} = 4.$$

That is, four corner patches, carrier rank minus one, matter-to-gauge compression factor, and root-to-start quotient all evaluate to the same integer.

9.2 Carrier compressions on the minimal branch

Corollary 9.5 (Two-point spectral compression of γ and φ_{base})**.** *Let*

$$y_+ := \frac{1}{2}, \qquad y_- := -\frac{1}{3}$$

be the two distinct carrier eigenvalues singled out by the factorized carrier polynomial. Then

$$\gamma = y_+ - y_-, \qquad \varphi_{\text{base}} = \frac{y_+ + y_-}{\pi}.$$

Here $y_+ + y_-$ is the unweighted Vieta sum of the two distinct carrier eigenvalues, not the trace on E.

Proof. By Equation (3), equivalently by the factorization

$$6Y^2 - Y - \mathbf{1} = (2Y - \mathbf{1})(3Y + \mathbf{1}),$$

the two distinct carrier eigenvalues are exactly $y_+ = 1/2$ and $y_- = -1/3$. Their difference is

$$y_+ - y_- = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = \gamma,$$

while their sum is

$$y_+ + y_- = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Since $\varphi_{\text{base}} = (1-\gamma)/\pi = 1/(6\pi)$, the claim follows. □

Proposition 9.6 (Dual-root generation of the hypercharge packet and admissible cusp set)**.** Let

$$y_+ = \frac{1}{2}, \qquad y_- = -\frac{1}{3}.$$

Then all one-family hypercharges are generated by the same two carrier roots:

$$\begin{aligned} Y(Q_L) &= y_+ + y_-, & Y(u^c) &= 2y_-, & Y(d^c) &= -y_-, \\ Y(L_L) &= -y_+, & Y(e^c) &= 2y_+, & Y(\nu^c) &= 0. \end{aligned}$$

Moreover the positive transport cusps satisfy

$$\left\{1, \frac{2}{3}, \frac{1}{3}\right\} = \{2y_+, -2y_-, 2(y_+ + y_-)\}.$$

Thus the carrier packet and the transport grammar are generated by the same two-point root data.

Proof. On the minimal carrier one has

$$Y = \text{diag}(y_-, y_-, y_-, y_+, y_+)$$

with respect to the split $E_3 \oplus E_2$. The one-family packet S^+ carries the sectors $Q_L, u^c, d^c, L_L, e^c, \nu^c$, and their hypercharges are the sums of the occupied carrier roots. This gives the displayed formulas. Substituting $y_+ = 1/2$ and $y_- = -1/3$ recovers the canonical values

$$\frac{1}{6}, \quad -\frac{2}{3}, \quad \frac{1}{3}, \quad -\frac{1}{2}, \quad 1, \quad 0.$$

The admissible transport cusps are fixed in [TFPT cross-reference: `thm:exact-transport-closure`] as $\{1, \frac{2}{3}, \frac{1}{3}\}$, which is exactly the displayed root-generated set. □

Corollary 9.7 (Family hypercharge variance and electroweak block trace)**.** *Equip one carrier family with the uniform average*

$$\mathbb{E}_{S^+}(A) := \frac{1}{16}\,\mathrm{Tr}_{S^+} A.$$

Then

$$\mathrm{Var}_{S^+}(Y) := \mathbb{E}_{S^+}(Y^2) - \mathbb{E}_{S^+}(Y)^2 = \frac{1}{16}\,\mathrm{Tr}_{S^+} Y^2 = \frac{5}{24}.$$

Consequently,

$$I_1^{\mathrm{EW}} = \mathrm{Var}_{S^+}(Y)\, b_1.$$

Proof. The one-family trace identities give $\mathrm{Tr}_{S^+} Y = 0$ and $\mathrm{Tr}_{S^+} Y^2 = 10/3$. Hence the centered second moment is

$$\frac{1}{16}\,\mathrm{Tr}_{S^+} Y^2 = \frac{10/3}{16} = \frac{5}{24}.$$

Using $I_1^{\mathrm{EW}} = \frac{5}{24}\, b_1$ from the electroweak block relation yields the second identity. □

Corollary 9.8 (Factorial quadratic trace and two-defect factorization)**.** *Let*

$$X := 6Y.$$

Then the one-family quadratic trace is

$$\mathrm{Tr}_{S^+}(X^2) = 36\ \mathrm{Tr}_{S^+}(Y^2) = 36 \cdot \frac{10}{3} = 120 = 5!.$$

Equivalently, using the six hypercharge sectors with multiplicities

$$(1, 3, 6, 1, 2, 3)$$

at

$$X \in \{0, -4, 1, 6, -3, 2\},$$

the same trace is

$$0^2 + 3 \cdot 4^2 + 6 \cdot 1^2 + 1 \cdot 6^2 + 2 \cdot 3^2 + 3 \cdot 2^2 = 120.$$

On the retained branch this gives the exact factorization

$$\delta_2 = 2880\, c_3^8 = 4!\ \mathrm{Tr}_{S^+}(X^2)\, c_3^8.$$

Proof. By [TFPT cross-reference: `prop:split-weight-enumerator`], one has

$$\mathrm{Tr}_{S^+}(Y^2) = \frac{10}{3}.$$

Since $X = 6Y$, the first identity follows immediately. The sector-by-sector sum is the same quadratic trace written in the basis of admissible occupancy sectors from [TFPT cross-reference: `prop:family-hypercharge-compression`]. Finally, the hard kernel normalization gives

$$\delta_2 = 2880\, c_3^8 = 24 \cdot 120\, c_3^8 = 4!\ \mathrm{Tr}_{S^+}(X^2)\, c_3^8.$$

□

Remark (Quadratic trace versus variance)*.* The value $120 = 5!$ is the unnormalized quadratic trace $\mathrm{Tr}_{S^+}(X^2)$, not the normalized variance. The corresponding uniform variance is

$$\mathrm{Var}_{S^+}(X) = 36\ \mathrm{Var}_{S^+}(Y) = 36 \cdot \frac{5}{24} = \frac{15}{2}.$$

Remark (Generic abelian-index identity). Define the generic occupancy

$$\Omega_{\rm occ} := 16\, N_{\rm fam}.$$

Then the abelian-index formula gives

$$10\, b_1 = \frac{40}{3} N_{\rm fam} + N_\Phi = \Omega_{\rm occ}\, \gamma + N_\Phi.$$

At this stage no canonical specialization is inserted yet. The closed 41 chain will be derived only after the family and Higgs theorems fix $\Omega_{\rm occ} = 48$ and $N_\Phi = 1$.

Corollary 9.9 (Family integrality repair)**.** *One carrier family contributes the weighted transport budget*

$$\dim S^+ \cdot \gamma = 16 \cdot \frac{5}{6} = \frac{40}{3}.$$

Hence any integral global budget assembled from whole carrier families must satisfy

$$N_{\rm fam} \cdot \frac{40}{3} \in \mathbb{Z}.$$

The minimal positive repair is therefore

$$N_{\rm fam,min} = 3.$$

For this minimal repair one gets

$$\Omega_{\rm occ} = N_{\rm fam,min} \dim S^+ = 48, \qquad \Omega_{\rm occ}\gamma = 40.$$

The later family theorem realizes exactly this minimal arithmetic.

Proof. The first identity uses $\dim S^+ = 16$ and $\gamma = 5/6$. Since $\gcd(40, 3) = 1$, integrality forces $3 \mid N_{\rm fam}$, so the smallest positive choice is 3. Multiplying by 16 and then by γ gives the displayed values. □

9.3 Generic abelian index identity

The abelian coefficient is first carried only in the generic form

$$10\, b_1 = \Omega_{\rm occ}\gamma + N_\Phi. \tag{13}$$

The arithmetic specialization to 41 is deferred until the family multiplicity and the Higgs sector have both been theorematically closed.

10 Closure architecture A: families, Higgs, and local spectral scale

This first closure-architecture block is the shortest route from the hard carrier packet to the closed theorem body. It has three moves: primitive winding balance in the seam normal plane forces the four-corner structure of the fundamental section, that winding theorem globalizes into the family space F, the determinant line selects the unique light Higgs doublet together with the compact seam phase, and the local spectral-scale channel generates both $\bar{M}_{\rm Pl}$ and the electroweak vacuum from one common geometric response.

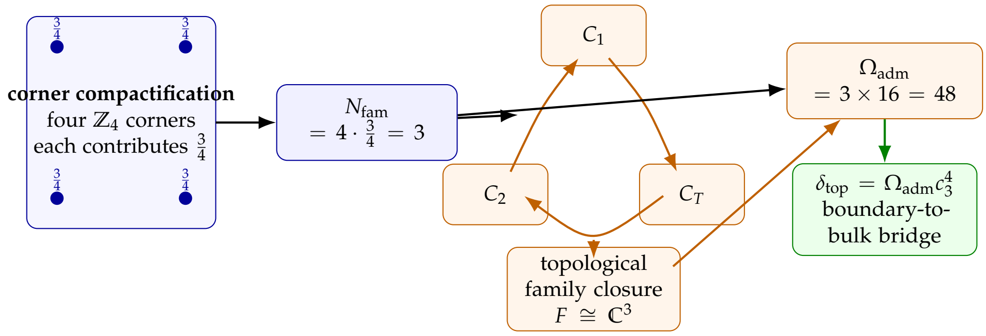


Figure 3. Carrier-to-family closure bridge. Corner compactification yields the punctured family section X_f°, the harmonic family space $\mathcal{H}_F \cong F \cong H^1(X_f^\circ, \mathbb{C})$ gives the hard multiplicity $N_{\text{fam}} = 3$, and the resulting occupancy $\Omega_{\text{adm}} = 48$ feeds back into the bridge reading $\delta_{\text{top}} = \Omega_{\text{adm}} c_3^4$.

10.1 Family monodromy and occupancy

Definition 10.1 (Family space)**.** Let $\overline{X}_f$ denote the compact seam quotient with four $\mathbb{Z}_4$ corner points $p_1, \dots, p_4$, and define the punctured family section

$$X_f^\circ := \overline{X}_f \setminus \{p_1, p_2, p_3, p_4\}.$$

Its rigid harmonic model is the Möbius class

$$X_f^\circ \cong \mathbb{P}^1 \setminus \mu_4.$$

Set

$$F := H^1(X_f^\circ, \mathbb{C}).$$

We use the distinguished cycle basis (C_1, C_2, C_T) only as a concrete carrier-side basis of this family space; any later APS/Hodge realization is read as compatible analytic structure on the same topological family space.

In this interpretation F is neither an arbitrary flavor vector space nor a loose bookkeeping trick; it is a cycle space attached to the seam data. The actual bridge logic is shown in Figure 3: the derived corner theorem fixes $N_{\text{fam}} = 3$, the topological family step promotes that count to a family space, and the occupancy corollary then feeds back into the defect channel.

The Hodge metric on F turns the monodromy data into an honest unitary family system rather than a merely decorative triangle sketch.

Theorem 10.2 (Four corners and the rigid family surface $\mathbb{P}^1 \setminus \mu_4$)**.** *Let $\overline{X}_f$ be the compact seam quotient induced by the deck involution τ_{dbl} and its spin lift on the oriented normal two-plane. Every admissible fixed corner contributes one quarter-turn of the lifted spin rotation. Therefore*

$$N_{\text{corner}} \cdot \frac{\pi}{2} = 2\pi, \qquad N_{\text{corner}} = 4.$$

Removing the four corner points gives a four-punctured sphere

$$X_f^\circ := \overline{X}_f \setminus \{p_1, p_2, p_3, p_4\}.$$

The inherited finite symmetry group on the puncture set is a faithful dihedral group $D_4 \subset PGL_2(\mathbb{C})$. Every faithful copy of D_4 in $PGL_2(\mathbb{C})$ is Möbius conjugate to the standard action generated by

$$r(z) = iz, \qquad s(z) = 1/z.$$

Up to Möbius equivalence the unique D_4-invariant four-point configuration is therefore

$$\mu_4 := \{z \in \mathbb{C} : \ z^4 = 1\} = \{1, -1, i, -i\}.$$

Hence

$$X_f^\circ \cong \mathbb{P}^1 \setminus \mu_4, \qquad \chi(X_f^\circ) = -2, \qquad H_1(X_f^\circ, \mathbb{Z}) \cong \mathbb{Z}^3.$$

Consequently

$$F := H^1(X_f^\circ, \mathbb{C}) \cong \mathbb{C}^3.$$

Proof. The quarter-turn count gives four corner points. A sphere with four removed points is isomorphic to $\mathbb{P}^1 \setminus \{q_1, q_2, q_3, q_4\}$ for some ordered quadruple. The classification of finite subgroups of $PGL_2(\mathbb{C})$ implies that every faithful dihedral subgroup of order eight is Möbius conjugate to the standard D_4 action generated by $z \mapsto iz$ and $z \mapsto 1/z$. Under that action the unique invariant orbit of cardinality four is the set of fourth roots of unity, namely $\mu_4 = \{1, -1, i, -i\}$. Therefore

$$X_f^\circ \cong \mathbb{P}^1 \setminus \mu_4.$$

The Euler characteristic is $2 - 4 = -2$, and for a genus-zero surface with four punctures one has rank $H_1 = 4 - 1 = 3$. Dualizing gives $F \cong \mathbb{C}^3$. □

Theorem 10.3 (Three harmonic family modes on the four-punctured family surface)**.** *Let*

$$X_f^\circ = \mathbb{P}^1 \setminus \mu_4$$

with a complete cusp metric in its Möbius class, and define the family Hilbert space by

$$\mathcal{H}_F := \ker \Delta^{(1)}_{L^2,\mathrm{APS}}(X_f^\circ).$$

Then

$$\dim \mathcal{H}_F = 3.$$

If the physical family multiplicity is defined by

$$N_{\mathrm{fam}} := \dim \mathcal{H}_F,$$

then

$$N_{\mathrm{fam}} = 3, \qquad F \cong \mathcal{H}_F \cong \mathbb{C}^3.$$

Proof. For a compact truncation $\overline{X}_{f,R}$ with four boundary circles,

$$\chi(\overline{X}_{f,R}) = 2 - 4 = -2.$$

Hence

$$\dim H^1(\overline{X}_{f,R}, \partial \overline{X}_{f,R}; \mathbb{C}) = 3.$$

The L^2 Hodge theorem for finite-area cusp surfaces identifies

$$\ker \Delta^{(1)}_{L^2,\mathrm{APS}}(X_f^\circ) \cong H^1(\overline{X}_{f,R}, \partial \overline{X}_{f,R}; \mathbb{C}),$$

so the dimension is 3. □

Remark (How the family theorem is now used)*.* The family theorem is used at rigid conformal strength in the present version. Any orbifold APS realization is read as compatible analytic structure on the same classification-rigid four-punctured section, while the physical multiplicity itself is carried by the harmonic family mode space $\mathcal{H}_F$.

Proposition 10.4 (Orbifold APS compatibility)**.** Under the additional analytical hypotheses of [TFPT cross-reference: `app:aps`],

$$\mathrm{Ind}^{\mathrm{orb,APS}}(D^+_{\mathrm{rel}} \otimes L_F) = 3.$$

Corollary 10.5 (Boundary-to-bulk bridge corollary for the admissible occupancy)**.** *On the canonical branch,*

$$\dim(S^+ \otimes \mathcal{H}_F) = 16 \cdot 3 = 48 = \Omega_{\mathrm{adm}}.$$

Hence the topological surcharge enters the present paper as a derived occupancy corollary:

$$\delta_{\mathrm{top}} = c_3^4 \dim(S^+ \otimes \mathcal{H}_F) = \Omega_{\mathrm{adm}}\, c_3^4.$$

The family subsection is therefore no longer split between a local count, an independent index theorem, and a later occupancy reinterpretation. Those statements are compressed into one derived corner theorem with one bridge corollary, while the holonomy proposition below records the resulting unitary geometry on $F \cong \mathcal{H}_F$.

This theorem is the family statement referenced later by the admissibility theorems: the closure layer uses it as the explicit family component of the master closure theorem rather than as an external import.

Proposition 10.6 (Rigid special-unitary family holonomy on the determinant-trivial branch)**.** Let $L_F \to X_f^\circ$ be the flat Hermitian family bundle with fiber $F \cong \mathcal{H}_F \cong H^1(X_f^\circ, \mathbb{C})$. Assume that its determinant line is flat trivial,

$$\det L_F \cong X_f^\circ \times \mathbb{C},$$

and that the chosen flat connection preserves this trivialization. Then

$$\mathrm{Hol}(\nabla_F) \subset SU(3)_F.$$

Proof. The Hermitian metric implies

$$\mathrm{Hol}(\nabla_F) \subset U(3).$$

The induced connection on $\det L_F$ has trivial holonomy by assumption, so for every loop γ one has

$$\det \rho_F(\gamma) = 1.$$

Hence every holonomy matrix lies in $SU(3)$. □

The role of the present subsection is thus no longer only mnemonic. It displays both the local patch-deficit count and the global APS/Hodge realization before the later closure stack compresses them into the theorem-level output surface.

10.2 Occupancy and boundary-to-bulk bridge

With the family bridge installed, the matter space becomes $S^+ \otimes \mathcal{H}_F$ and the occupancy $\Omega_{\mathrm{adm}} = 48$ is already fixed by the harmonic family-mode theorem and its bridge corollary. Here *occupancy* means the admissible chiral-state count after family lifting; it is not used as a primitive datum. The bridge point is not that family lifting creates a new numerical surcharge. Rather, the inherited hard kernel value

$$\delta_{\mathrm{top}} = \frac{3}{256\pi^4}$$

admits on the canonical family branch the occupancy reinterpretation $\delta_{\rm top} = \Omega_{\rm adm}\, c_3^4$. Likewise the hard second-order defect contribution is re-read as

$$\delta_2 = \frac{5}{4}\,\delta_{\rm top}^2 = B\gamma\,\Omega_{\rm adm}^2 c_3^8.$$

In the present manuscript this occupancy corollary is used with a derived bridge reading: it is the boundary-to-bulk step that later feeds both the electromagnetic closure and the geometric channel. The coefficient 5/4 is therefore not an extra numerical motif; it is the minimal-carrier compression factor $B\gamma$ rewritten in defect language. At the same time, the equality $\delta_{\rm top} = \Omega_{\rm adm} c_3^4$ is not presented as an independent dynamical transgression theorem in the ordinary field-theory sense.

10.3 From bridge to geometric completion

Principle 10.7 (Closure split)**.** Once the carrier packet and family closure architecture are fixed, the focused main-text closure problem splits into a geometric program $\mathcal{A}_{\rm geom}$ and a transport program $\mathcal{A}_{\rm trans}$. Record algebra and prediction semantics are now tied directly to the admissible gap in the main theorem stack, while E_8 scale grammar and horizon extrapolations are deferred to appendix-level continuations.

$$D_{\rm geo} := D_{\rm ref} + D_{\rm rel} = D.$$

The two main-text programs are packaged as

$$\mathcal{A}_{\rm geom} := \langle E_3 \oplus E_2,\ D_{\rm geo},\ \mathcal{B}_{\rm rel},\ \varpi_{\rm spin}\rangle, \qquad \mathcal{A}_{\rm trans} := \langle U_6,\ T,\ D_y,\ \mathcal{Y}_y^{(\varepsilon)},\ \mathfrak{C}_\Sigma\rangle.$$

This split keeps the carrier theorem, the geometric reconstruction problem, and the record-level extensions from being flattened into one undifferentiated claim. In particular, the metric field is no longer treated here as an independent entry of $\mathcal{A}_{\rm geom}$; it is reconstructed from the geometric Dirac anchor introduced next, while $D_{\rm rel}$ remains the comparison operator for spectral subtraction.

10.4 Relative APS and superconnection setup

The geometric completion is formulated on a graded bundle

$$\mathcal{E}_{\rm rel} = \mathcal{E}_{\rm rel}^+ \oplus \mathcal{E}_{\rm rel}^-,$$

with the geometric Dirac anchor $D_{\rm geo}$, the declared reference operator $D_{\rm ref}$, and APS-type boundary conditions on the seam Σ; see Appendix [TFPT cross-reference: `app:aps`]. The geometric package is

$$\mathbb{A}_\Sigma := (D_{\rm geo}, D_{\rm rel}, A_\Sigma^{\rm geo}), \qquad A_\Sigma^{\rm geo} := \mathcal{B}_{\rm rel} \oplus \nabla_{\chi_{\rm geo}}.$$

Here $D_{\rm geo}$ is the geometric Dirac anchor, while $D_{\rm rel}$ is the relative comparison operator entering the spectral subtraction formulas. The bundle data $A_\Sigma^{\rm geo}$ reorganize the primitive seed package $A_\Sigma^{\rm seed}$ introduced earlier in [TFPT cross-reference: `eq:tfpt-master`].

Theorem 10.8 (Geometric reconstruction from the Dirac anchor)**.** *Assume*

$$Z(\mathcal{A}) \cong C^\infty(M),$$

and the geometric anchor $D_{\rm geo}$ is of Dirac type with principal symbol

$$\sigma(D_{\rm geo})(x,\xi)^2 = g^{\mu\nu}(x)\xi_\mu\xi_\nu\,\mathbf{1}.$$

Then D_{geo} reconstructs the metric g, the spin structure, and the Levi–Civita connection on the geometric branch. Inner fluctuations of the same anchored spectral datum generate the gauge sector. The operator D_{rel} is the reference-subtracted comparison operator on that same branch and is not itself used as the local geometric anchor.

Proof sketch. On the admissible branch the commutative center identifies the manifold algebra, while the principal symbol of the Dirac anchor determines the quadratic form on cotangent vectors. Standard Dirac-type reconstruction then recovers the spin bundle and Levi–Civita connection from that same symbol data. The relative operator changes the comparison bookkeeping, not the geometric source of the local symbol. □

Theorem 10.9 (Retained branch from the admissibility selector)**.** *Let D_{geo} be the geometric anchor and let P_{adm} be the retained admissibility selector. Then:*

(i) $X_{\mathrm{bulk}} = \operatorname{Spec} Z(\mathcal{A})$ *and* $\Sigma = \operatorname{Fix}(\tau_{\mathrm{dbl}})$*;*

(ii) the local spectral density of D_{geo} determines the geometric response χ_{geo};

(iii) P_{adm} defines the retained admissible sector for observables, transport, and positivity statements on that branch.

No claim is made that the compressed operator $P_{\mathrm{adm}} D_{\mathrm{rel}} P_{\mathrm{adm}}$ carries the local principal symbol of the geometry.

Corollary 10.10 (No residual geometric datum beyond the geometric anchor)**.** *The geometric package is reconstructed from D_{geo} and its local spectral density. The selector P_{adm} acts downstream on the retained sector and is not part of the local geometric anchor.*

Definition 10.11 (Local spectral scale and canonical relative spectral residues)**.** Write the geometric response as a local spectral scale

$$\chi_{\mathrm{geo}} = \Lambda e^{\sigma}, \qquad D_{\sigma,\mathrm{geo}} := e^{-\sigma/2} D_{\mathrm{geo}} e^{-\sigma/2}.$$

$$\mathcal{Z}_{\mathrm{rel}}(s;\sigma) := \operatorname{Tr}_{\mathrm{rel}}\big(|D_{\sigma,\mathrm{geo}} + \mathcal{B}_{\mathrm{rel}}|^{-2s} - |D_{\mathrm{ref}}|^{-2s}\big) + \frac{i\pi}{2}\Delta\eta_{\Sigma}.$$

The canonical Einstein residue and order-zero residue are

$$\mathcal{E}_D := \operatorname*{Res}_{s=1} \mathcal{Z}_{\mathrm{rel}}(s;\sigma), \qquad \Gamma_D^{(4)} := \mathrm{FP}_{s=0}\, \mathcal{Z}_{\mathrm{rel}}(s;\sigma).$$

The gravitational branch is then read from the profile-free residue pair rather than from a freely chosen spectral profile.

Corollary 10.12 (Constant-scale reduction of the geometric branch)**.** *If the local spectral scale is frozen to a constant $\sigma = \sigma_0$, then $\chi_{\mathrm{geo}} = \chi_{\mathrm{geo},0} := \Lambda e^{\sigma_0}$ and the present geometric branch reduces to the constant-χ_{geo} closure relations extracted from $\Gamma_{\mathrm{grav}} := -6\chi_{\mathrm{geo}}^2 \mathcal{E}_D + \Gamma_D^{(4)}$:*

$$\bar{M}_{\mathrm{Pl}}^2 = \frac{\chi_{\mathrm{geo},0}^2}{2\pi^2}, \qquad v_{\mathrm{geo}}^2 = Z_H\, \frac{\mu_{\Phi}^2(\chi_{\mathrm{geo},0})}{\lambda_{\Phi}}.$$

Once the unique stationary root of [TFPT cross-reference: `thm:absolute-spectral-planck-closure`] *is imposed, the Einstein coefficient is equivalently the boundary-normalized readout*

$$\frac{\bar{M}_{\mathrm{Pl}}^2}{\lambda_{\Sigma}^2} = \frac{\rho_\star}{2\pi^2}.$$

Thus the former constant-χ_{geo} formulas are retained as the homogeneous branch of the local spectral-scale formulation rather than as a competing package.

10.5 Bosonic relative index and Higgs selection

Theorem 10.13 (Minimal determinant classes from the primitive charge generator)**.** *Let*

$$L_2 := \det E_2, \qquad L_3 := \det E_3$$

over the compactified oriented normal sphere $S^2 = D_N \cup_{S^1} D_S$*. Then the primitive charge generator*

$$X = -2P_3 + 3P_2$$

together with center neutrality and seam-evenness forces the clutching map

$$g_2(\varphi) = e^{i\varphi}$$

for L_2*, while center neutrality forces*

$$g_3(\varphi) = 1$$

for L_3*. Therefore*

$$c_1(L_2) = \deg g_2 = 1, \qquad c_1(L_3) = \deg g_3 = 0.$$

This pair is the unique minimal nonnegative admissible determinant class.

Proof. Complex line bundles on

$$S^2 = D_N \cup_{S^1} D_S$$

are classified by their clutching maps along the equator:

$$\pi_1(U(1)) \cong \mathbb{Z}.$$

Hence the first Chern class is exactly the winding degree of the transition function.

In the weak rank-two sector the primitive charge generator

$$X = -2P_3 + 3P_2$$

assigns the unique positive determinant winding to the seam-even weak block. The spin lift on the oriented normal sphere acts there by the half-angle representation, so on the determinant line the transition factor is

$$g_2(\varphi) = e^{i\varphi},$$

hence

$$\deg g_2 = 1, \qquad c_1(L_2) = 1.$$

In the color sector admissibility imposes center neutrality. Therefore the determinant transition function of the color block is null-homotopic and may be chosen constant:

$$g_3(\varphi) = 1.$$

Consequently

$$\deg g_3 = 0, \qquad c_1(L_3) = 0.$$

The full primitive seam twist is therefore exhausted in the weak determinant class. Any nonnegative pair strictly smaller than $(1, 0)$ would force vanishing weak determinant winding and hence remove the seam-even weak line required by the primitive charge generator. Any larger nonnegative pair would insert extra clutching not carried by the primitive generator and therefore violate minimality. Thus the unique minimal nonnegative admissible determinant class is

$$(c_1(L_2), c_1(L_3)) = (1, 0).$$

□

Remark (Energy interpretation on the retained branch). The older asymptotic bosonic functional

$$\mathcal{E}_{\mathrm{asym}}(\Phi_2, \Phi_3) := \tau_2 \big|\mathrm{wind}(\det \Phi_2)\big| + \tau_3 \big|\mathrm{wind}(\det \Phi_3)\big| + m_3^2 \dim \ker \mathcal{B}^+_{E_3},$$

with $\tau_3 > \tau_2 > 0$, remains a useful physical intuition for why the retained branch is energetically preferred. In the present rewrite, however, the structural weight is carried by the determinant-class theorem above rather than by the cost functional itself.

Theorem 10.14 (Bosonic index and unique Higgs doublet from the minimal determinant class)**.** *Compactify the oriented two-dimensional normal slice N_Σ to S^2. For $a \in \{2, 3\}$ define*

$$\mathcal{B}_{E_a} := \sigma^\mu \nabla^{(a)}_\mu + \Phi_a$$

on the compactified normal sphere. Under the determinant-class data of Theorem 10.13, one has

$$L_2 \cong \mathcal{O}(1), \qquad L_3 \cong \mathcal{O},$$

and hence

$$H^0(S^2, \mathcal{O}(1)) \cong \mathbb{C}^2, \qquad H^1(S^2, \mathcal{O}(1)) = 0,$$

and the compact bosonic index satisfies

$$\mathrm{Ind}(\mathcal{B}^+_{E_2}) = 1, \qquad \mathrm{Ind}(\mathcal{B}^+_{E_3}) = 0.$$

Consequently the closed branch contains exactly one complex weak doublet and no light color triplet, and the unique light seam-even bosonic zero mode lies in

$$\Phi \in \Gamma(E_2) \cong (1, 2)_{1/2}, \qquad \tau^*_{\mathrm{dbl}} \Phi = +\Phi.$$

Proof. By Theorem 10.13, the weak determinant line is the degree-one line bundle

$$L_2 \cong \mathcal{O}(1)$$

on

$$S^2 \cong \mathbb{C}P^1,$$

whereas the color determinant class is trivial and

$$L_3 \cong \mathcal{O}.$$

For $\mathbb{C}P^1$, Birkhoff–Grothendieck identifies every holomorphic line bundle by its degree, and Riemann–Roch gives

$$\chi(\mathcal{O}(1)) = 1 + 1 = 2.$$

Serre duality yields

$$H^1(S^2, \mathcal{O}(1)) \cong H^0(S^2, \mathcal{O}(-3))^\vee = 0,$$

so

$$H^0(S^2, \mathcal{O}(1)) \cong \mathbb{C}^2, \qquad H^1(S^2, \mathcal{O}(1)) = 0.$$

The resulting two-dimensional zero-mode space is exactly one complex weak doublet, so the compact bosonic index records it as

$$\mathrm{Ind}(\mathcal{B}^+_{E_2}) = 1.$$

In the color sector one has $L_3 \cong \mathcal{O}$, so no positive determinant degree is available. Center neutrality forbids an unpaired seam-even color contribution, hence the positive and negative triplet modes pair with zero net index:

$$\mathrm{Ind}(\mathcal{B}^+_{E_3}) = 0.$$

Therefore the closed branch contains exactly one seam-even Higgs doublet and no light color triplet, and the unique light bosonic zero mode lies in the weak block. □

Corollary 10.15 (Bosonic rank is two). *On the closed branch the positive carrier block has rank*

$$\dim E_+ = 2.$$

Equivalently, the unique seam-even light bosonic zero mode fills one complex weak doublet and no larger positive carrier block is retained.

Proof. By [TFPT cross-reference: `thm:callias-higgs-selection`], the light seam-even bosonic zero mode lies in the weak block

$$\Phi \in \Gamma(E_2) \cong (1,2)_{1/2},$$

and the compact index identifies its zero-mode space with

$$H^0(S^2, \mathcal{O}(1)) \cong \mathbb{C}^2.$$

Hence the retained positive carrier block is exactly two-dimensional. □

This is the sharp form of the Higgs claim used here. The issue is not whether a Higgs doublet can be written down, but whether a compact bosonic index can be shown to select it uniquely.

Remark (Callias shadow). The older Callias formulation is retained only as the noncompact shadow of the same compact index theorem and is discussed separately in the appendical analytic remarks. It no longer carries the main proof burden.

Once that uniqueness statement is fixed, the renormalizable carrier-compatible Yukawa couplings are the standard ones,

$$Q\,u^c\,\Phi, \qquad Q\,d^c\,\Phi^\dagger, \qquad L\,e^c\,\Phi^\dagger, \qquad L\,\nu^c\,\Phi,$$

so the Higgs statement becomes a genuine structural bridge rather than a notation choice. The formal carrier discharge from this Yukawa bridge is recorded later in [TFPT cross-reference: `cor:bosonic-rank-two,thm:yukawa-forces-32,cor:yukawa-discharges-kpp`], once the closure-theorem Higgs block has been stated in its final retained-branch form.

The two-stage selector geometry is summarized in Figure 4.

Theorem 10.16 (Calderón upgrade of the APS boundary domain). *Let* $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ *be the provisional APS realization of the geometric anchor used in* [TFPT cross-reference: `thm:well-posed-primitive-dynamics`], *and let* D_b *be the canonical doubling of* D_{rel} *across* Σ. *Let* $C(D_b)$ *denote the Calderón projector of* D_b *on the seam collar. Then there is a canonical boundary-elliptic upgrade*

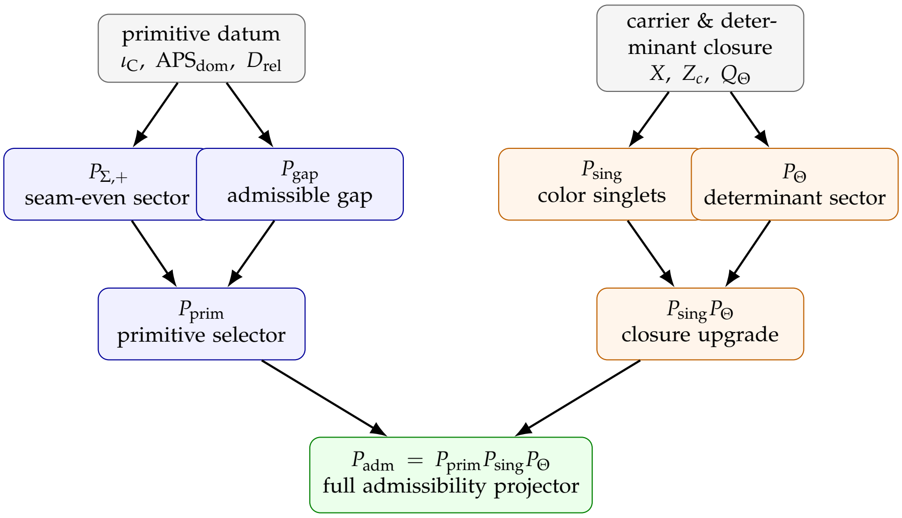
$$\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}} \xrightarrow{C(D_b)} \mathrm{APS}_{\mathrm{dom}}$$

such that:

(i) $\mathrm{APS}_{\mathrm{dom}}$ *lies in the same admissible elliptic class as* $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ *and preserves the self-adjoint Fredholm realization of the geometric anchor;*

(ii) the relative graph domain of D_{rel} *is unchanged, so all primitive results that depended only on* $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ *continue to hold with* $\mathrm{APS}_{\mathrm{dom}}$*;*

(iii) the family-pairing data $\mathrm{APS}_{\mathrm{fam}}$ *are induced canonically on the doubled admissible branch and are not a separate primitive datum.*



Primitive data determine P_{prim}; carrier and determinant closure upgrade it to P_{adm}.

Figure 4. Two-stage upgrade from the primitive selector to the full admissibility projector. The left lane is fixed by the primitive datum, while the right lane enters only after carrier and determinant closure.

Proof sketch. The Calderón projector $C(D_b)$ is canonically associated with the doubled operator on the seam collar and produces a boundary-elliptic projection compatible with the provisional APS choice. Both APS^{prov}_{dom} and APS_{dom} are admissible elliptic boundary conditions for the same Dirac-type operator and therefore define equivalent self-adjoint Fredholm realizations of the geometric anchor (boundary-elliptic equivalence preserves the Sobolev graph domain). Consequently every statement made on the primitive scaffold under APS^{prov}_{dom} persists on the closed branch under APS_{dom}. The induction of APS_{fam} from canonical doubling is a standard consequence of the cobordism structure of the doubled APS pair. □

Proposition 10.17 (Master closure roadmap from canonical doubling)**.** This proposition is a roadmap statement: it lists the closure conclusions that the individual theorems of this section establish from $\mathfrak{T}^{min}_\partial$, the derived closed datum, the carrier theorem, and the canonical doubling. It is not used as an input to those proofs and is recovered as a master corollary at the end of the section. Let X_b be the canonical double across Σ and write

$$D_b = \begin{pmatrix} 0 & D^-_{rel} \\ D^+_{rel} & 0 \end{pmatrix}, \qquad Q_{tot} := Q_{adm} + Q_{geo}, \qquad \Delta_{tot} := \{Q_{tot}, Q^\dagger_{tot}\}.$$

Then on the closed branch all of the following are derived from the same datum (each item is the output of a downstream theorem in this section, not an extra hypothesis here):

(1) the Calderón projector of D_b canonically upgrades the provisional APS choice APS^{prov}_{dom} to APS_{dom} (cf. [TFPT cross-reference: `thm:apsdom-upgrade`]);

(2) the family pairing data are induced canonically, so no separate APS_{fam} datum remains;

(3) the upgrade theorem [TFPT cross-reference: `thm:padm-upgrade`] produces

$$P_{adm} := \Pi_{\ker \Delta_{adm}} = P_{prim}\, P_{sing}\, P_\Theta,$$

and the doubled total closure package preserves $\mathrm{Ran}(P_{adm})$;

(4)

$$\chi(X^\circ_f) = -2, \qquad F \simeq H^1(X^\circ_f, \mathbb{C}) \simeq \mathbb{C}^3, \qquad N_{fam} = 3;$$

(5) the harmonic determinant line is parallel and therefore

$$\mathrm{Hol}(\nabla_F) \subset SU(3)_F;$$

(6) admissible bosonic, fermionic, and geometric reflection positivity hold on $\mathrm{Ran}(P_{\mathrm{adm}})$;

(7) the physical geometric sector takes the form

$$\mathcal{H}_{\mathrm{phys}} \cong \mathcal{H}_2 \otimes \mathcal{H}_{\mathrm{sc}} \otimes \mathcal{H}_{\mathrm{matt}},$$

with the scalaron carried by the positive $(\chi_{\mathrm{geo}}, R^2)$ block;

(8) $P_{\mathrm{adm}}\mathcal{H}_{\mathrm{rel}}P_{\mathrm{adm}}$ is lower bounded and coercive.

Definition 10.18 (Relative generating functional)**.** The relative Euclidean generating functional is normalized as

$$Z_{\mathrm{rel}}[J,\eta,\bar{\eta}] := \frac{Z_{\mathrm{adm}}[J,\eta,\bar{\eta}]}{Z_{\mathrm{ref}}[0,0,0]}.$$

The reference sector is therefore a fixed normalizing denominator rather than a second fluctuating measure that would mix with admissible positivity.

Lemma 10.19 (Admissible reflection positivity)**.** *Assume*

$$[\Theta, P_\Theta] = 0, \qquad \Theta P_{\mathrm{adm}} = P_{\mathrm{adm}}\Theta,$$

the admissible bosonic kernel satisfies the Bernstein-positivity remark above and is reflection-stable on the admissible sector, and the admissible fermionic CAR kernel is reflection positive on that same sector. Then every admissible observable O supported in $\tau > 0$ satisfies

$$\langle \Theta O, O\rangle_{\mathrm{rel}} = \frac{1}{Z_{\mathrm{ref}}[0,0,0]}\langle \Theta O, O\rangle_{\mathrm{adm}} \geq 0.$$

Proof. By definition,

$$\langle \Theta O, O\rangle_{\mathrm{rel}} = \frac{1}{Z_{\mathrm{ref}}[0,0,0]}\,\langle \Theta O, O\rangle_{\mathrm{adm}}.$$

The denominator is a fixed positive normalizing constant, so the sign is determined entirely by the admissible numerator.

Split O into its bosonic and fermionic parts on the retained admissible sector. For the bosonic part, the Bernstein representation of the admissible kernel writes the Euclidean covariance as a positive superposition of reflection-stable heat kernels. Reflection across $\tau = 0$ therefore sends every positive-time bosonic observable to a positive quadratic form, so the bosonic contribution to

$$\langle \Theta O, O\rangle_{\mathrm{adm}}$$

is nonnegative.

For the fermionic part, the CAR reflection-positivity hypothesis on the admissible subspace implies that the reflected two-point kernel is positive in the standard fermionic sense; equivalently, the corresponding Pfaffian / determinant form on positive-time test spinors is nonnegative. Because the bosonic and fermionic admissible factors commute inside the normalized expectation, their product remains nonnegative. Dividing by the fixed reference normalization preserves that sign. Hence

$$\langle \Theta O, O\rangle_{\mathrm{rel}} \geq 0.$$

□

Theorem 10.20 (Operative admissibility selector)**.** *Under* [TFPT cross-reference: `def:full-admissibility-comp` *the operator P_{adm} is idempotent on the retained branch and realizes the admissibility predicate on all sectors used by the closed output statements:*

$$\mathsf{Adm}(X) = 1 \iff P_{\mathrm{adm}} X = X.$$

Proof sketch. By [TFPT cross-reference: `thm:padm-upgrade`], $P_{\mathrm{adm}} = P_{\mathrm{prim}} P_{\mathrm{sing}} P_{\Theta}$ is the orthogonal projector onto the harmonic admissible sector of the full complex and is therefore idempotent by construction. [TFPT cross-reference: `cor:factorized-admissibility-selector`] recovers the explicit factorized selector. The predicate and operator languages agree once the positive transport sector is fixed. □

Theorem 10.21 (Topological family closure on the admissible branch)**.** *By* [TFPT cross-reference: `thm:master-closure,thm:topological-family-closure,cor:boundary-bulk-occupancy`],

$$\chi(X_f^{\circ}) = -2, \qquad F \cong \mathbb{C}^3, \qquad N_{\mathrm{fam}} = 3.$$

Consequently

$$\Omega_{\mathrm{adm}} = N_{\mathrm{fam}} \dim S^{+} = 3 \times 16 = 48, \qquad \delta_{\mathrm{top}} = \Omega_{\mathrm{adm}}\, c_3^4.$$

Proof sketch. This theorem is now a direct consequence of the master closure theorem from Section 6. The topological family closure and the orbifold corner count have already been compressed there into one statement, so the present layer merely records the corresponding admissibility consequence. □

Theorem 10.22 (Bosonic index and local spectral-scale closure)**.** *On the admissible branch,*

$$\mathrm{Ind}_{\mathrm{rel}}^{P_{\mathrm{adm}}}(\mathcal{B}_{E_2}^{+}) = 1, \qquad \mathrm{Ind}_{\mathrm{rel}}^{P_{\mathrm{adm}}}(\mathcal{B}_{E_3}^{+}) = 0,$$

so the unique light seam-even bosonic zero mode is

$$\Phi \in \Gamma(E_2) \cong (1,2)_{1/2}, \qquad \tau_{\mathrm{dbl}}^{*}\Phi = +\Phi.$$

Moreover the same closure layer yields

$$\frac{\bar{M}_{\mathrm{Pl}}^2}{\lambda_\Sigma^2} = \frac{\rho_\star}{2\pi^2}, \qquad\qquad G_N \lambda_\Sigma^2 = \frac{\pi}{4\rho_\star},$$

$$\frac{v_{\mathrm{geo}}}{\bar{M}_{\mathrm{Pl}}} = g_{\mathrm{car}}\, \beta_{\mathrm{rad}}^2 \exp\left[-\frac{\alpha^{-1}(0) + \delta_{\mathrm{ph}}}{5}\right],$$

$$G_N v_{\mathrm{geo}}^2 = \frac{1}{8\pi} g_{\mathrm{car}}^2 \beta_{\mathrm{rad}}^4 \exp\left[-\frac{2(\alpha^{-1}(0) + \delta_{\mathrm{ph}})}{5}\right].$$

Here v_{geo} is the seam-even UV source scale. The physical electroweak benchmark is the residue matched quantity v_{phys} of [TFPT cross-reference: `thm:ew-geometric-to-physical-matching`].

Proof sketch. The bosonic sector is treated as a genuine Callias / Dirac / Dolbeault-type index problem rather than a scalar counting exercise. Relative heat-kernel closure together with [TFPT cross-reference: `prop:canonical-bosonic-normalization,lem:two-sheet-zero-mode-normalization,thm:a` produces the boundary-normalized Einstein branch, the canonical Higgs normalization, and the seam-even zero-mode formula for v_{geo}. □

Definition 10.23 (Primitive Yukawa generator)**.** Let

$$\mathcal{I}_Y := \bigoplus_{p,q,r,t} \mathrm{Hom}_{G_{\mathrm{car}}}\Big((\Lambda^p E_- \otimes \Lambda^q E_+) \otimes (\Lambda^r E_- \otimes \Lambda^t E_+) \otimes E_+, \mathbb{C} \Big)$$

be the graded algebra of carrier-invariant Yukawa trilinears. A homogeneous nonzero invariant is called primitive if it is indecomposable, equivalently if it does not lie in the ideal generated by positive-degree invariants of strictly lower carrier degree.

Lemma 10.24 (Lowest primitive Yukawa generator on the closed branch)**.** *Assume* $\dim E_+ = 2$ *and let* $\mathbb{Y}_{\mathrm{br}}$ *be the branch Yukawa tensor of* [TFPT cross-reference: `thm:exact-transport-closure,thm:rigid-fam` *Then the primitive generator of* $\mathcal{I}_Y$ *with two nontrivial fermionic legs is unique up to exchange of the two fermionic legs and belongs to*

$$\mathrm{Hom}_{G_{\mathrm{car}}}\big((E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+, \mathbb{C} \big).$$

Proof. The positive-block determinant neutrality gives $q + t + 1 = 2$, hence $q + t = 1$. Up to exchanging the two fermionic legs, $(q, t) = (1, 0)$. Evenness of both fermionic legs then forces p odd and r even. If $p \geq 3$, the corresponding invariant factors through an extra negative spectator wedge and therefore lies in the ideal generated by lower carrier degree. Likewise, if $r \geq 4$, the invariant factors through a spectator bivector insertion and is again decomposable. Primitive indecomposability therefore forces $p = 1$ and $r = 2$. □

Theorem 10.25 (Carrier rigidity from the primitive component of the branch Yukawa tensor)**.** *Let*

$$E = E_- \oplus E_+$$

be the two-factor carrier and let

$$G_{\mathrm{car}} := S\big(U(E_-) \times U(E_+)\big), \qquad S^+ = \Lambda^{\mathrm{even}} E.$$

Let $\mathbb{Y}_{\mathrm{br}}$ *denote the branch Yukawa tensor induced by the canonical transport kernel and rigid family holonomy of* [TFPT cross-reference: `thm:exact-transport-closure,thm:rigid-family-local-system`]. *Decompose its carrier-homogeneous components as trilinear maps of the form*

$$B_{p,q;r,t} : (\Lambda^p E_- \otimes \Lambda^q E_+) \otimes (\Lambda^r E_- \otimes \Lambda^t E_+) \otimes E_+ \longrightarrow \mathbb{C}.$$

Let the primitive component of $\mathbb{Y}_{\mathrm{br}}$ *be the unique indecomposable generator of* Definition 10.23 *and* lemma 10.24 *with two nontrivial fermionic legs. Then, up to exchanging the two fermionic legs, that primitive component is necessarily of type*

$$(E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+ \longrightarrow \Lambda^3 E_- \otimes \Lambda^2 E_+ \cong \mathbb{C},$$

and therefore

$$\dim E_+ = 2, \qquad \dim E_- = 3.$$

Consequently

$$\dim \Lambda^2 E_+ = 1, \qquad \Lambda^2 E_- \cong E_-^\vee, \qquad \dim \Lambda^{\mathrm{even}} E = 16.$$

Hence the rigid carrier is

$$E = E_3 \oplus E_2, \qquad X = -2P_3 + 3P_2, \qquad Y = -\frac{1}{3}P_3 + \frac{1}{2}P_2.$$

Proof. By [TFPT cross-reference: `cor:bosonic-rank-two`], the compact Higgs index has already fixed

$$\dim E_+ = 2.$$

Write the primitive nonzero Yukawa component with two nontrivial fermionic legs as

$$B_{p,q;r,t} : (\Lambda^p E_- \otimes \Lambda^q E_+) \otimes (\Lambda^r E_- \otimes \Lambda^t E_+) \otimes E_+ \longrightarrow \mathbb{C}$$

and note that Lemma 10.24 forces primitiveness in the invariant-ring sense to have

$$(q,t) = (1,0), \qquad (p,r) = (1,2)$$

up to exchanging the two fermionic legs. Thus the primitive carrier component is exactly

$$(E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+.$$

Its negative factor contracts canonically as

$$E_- \otimes \Lambda^2 E_- \longrightarrow \Lambda^3 E_-,$$

so scalarity forces $\Lambda^3 E_-$ to be the top exterior power. Hence

$$\dim E_- = 3.$$

With

$$(\dim E_-, \dim E_+) = (3,2),$$

one obtains the carrier-signature consequences

$$\Lambda^2 E_- \cong E_-^\vee, \qquad \dim \Lambda^2 E_+ = 1,$$

and therefore

$$\dim \Lambda^{\mathrm{even}}(E_3 \oplus E_2) = 2^{5-1} = 16.$$

Finally the unimodular primitive two-point generator is fixed by

$$3q_- + 2q_+ = 0, \qquad q_- < 0 < q_+, \qquad \gcd(q_-, q_+) = 1,$$

so $(q_-, q_+) = (-2, 3)$ and hence

$$X = -2P_3 + 3P_2, \qquad Y = \frac{X}{6} = -\frac{1}{3}P_3 + \frac{1}{2}P_2.$$

□

Lemma 10.26 (Uniqueness of the cubic weak invariant). *Let $E = E_- \oplus E_+$ be a two-factor carrier on the retained branch and let*

$$\Phi \in E_+$$

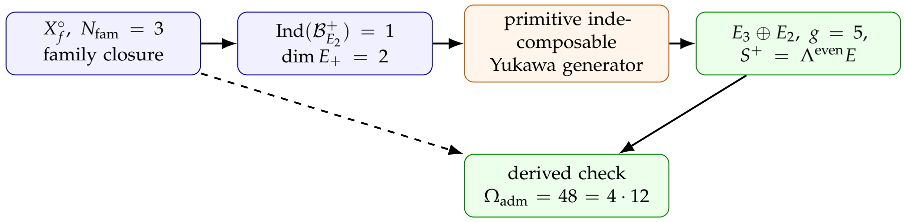
be the unique seam-even light boson selected by the compact bosonic index. Then

$$\dim \mathrm{Hom}_{G_{\mathrm{car}}}\big((E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+, \mathbb{C}\big) = 1$$

if and only if

$$(\dim E_-, \dim E_+) = (3,2).$$

In particular, on the retained branch the up-type cubic invariant is not an extra choice but the unique admissible cubic invariant.



Carrier closure runs through Higgs rank and branch Yukawa rigidity; the old occupancy formula is only a downstream consistency identity.

Figure 5. Carrier closure after the proof-tree cut. The retained branch first fixes $\dim E_+ = 2$ through the compact Higgs index, then $\dim E_- = 3$ through the primitive indecomposable generator of the actual branch Yukawa tensor; only afterwards is $\Omega_{\text{adm}} = 48 = 4 \cdot 12$ read as a consistency check.

Proof. By Schur functor decomposition,

$$\text{Hom}_{G_{\text{car}}}\big((E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+, \mathbb{C}\big) \cong \text{Hom}_{SU(E_-)}(E_- \otimes \Lambda^2 E_-, \mathbb{C}) \otimes \text{Hom}_{SU(E_+)}(E_+ \otimes E_+, \mathbb{C}).$$

The first factor is nonzero iff $\Lambda^2 E_- \cong E_-^\vee$, hence iff $\dim E_- = 3$. The second factor is nonzero iff the alternating contraction on two fundamentals is one-dimensional, hence iff $\dim E_+ = 2$. In that case both factors are one-dimensional, so the total invariant space is one-dimensional. □

Corollary 10.27 (Carrier-signature premises are discharged internally)**.** *Combine* [TFPT cross-reference: `cor:bosonic-rank-two,thm:yukawa-forces-32,lem:unique-cubic-weak-invariant`] *with the one-family decomposition of $S^+ = \Lambda^{\text{even}}(E_3 \oplus E_2)$ and with the unique seam-even Higgs theorem. Then the weak antisymmetric line, the color-dual antisymmetric sector, and the retained light-packet identification are no longer primitive inputs once the uniqueness of the retained cubic invariant has been proved on the same branch.*

Corollary 10.28 (Hard holonomy form of the Yukawa matrices)**.** *On the admissible branch, each fermion sector obeys the canonical factorization*

$$Y_f = D_{L,f} U_f^\star D_{R,f}, \qquad U_f^\star \in SU(3)_F,$$

and, in the family basis fixed by [TFPT cross-reference: `thm:exact-transport-closure`],

$$(Y_f)_{ij} = \lambda_Y^{L^L_{f,i} + L^R_{f,j}} \left\langle e_i, \text{Hol}_{\Gamma^{\min}_{ij}}(\nabla^\star_F) e_j \right\rangle.$$

Any factorization by $U_f^\star$ is therefore a consequence of the hard holonomy closure, not an independent flavor input. The diagonal path-length rows [TFPT cross-reference: `eq:path-length-yukawa`] *are the singular-value reduction of this matrix statement.*

Proof sketch. The finite C_6 transport geometry fixes the graph-distance weights, while the rigid D_4-equivariant family connection fixes the holonomy factors on the canonical path frame. The diagonal ladder formulas arise once left and right admissible words are paired on the same singular-value channel, so the matrix factorization is recovered as a corollary of one closed holonomy statement rather than as a separate ansatz. □

Definition 10.29 (Center-neutral admissible algebra)**.** Let

$$\mathcal{A}_{\text{sing}} := \overline{\text{span}}\{A_w : \ z(w) = 1,\ \varepsilon(w) = +1,\ P_{\text{adm}} A_w = A_w\}.$$

Theorem 10.30 (Hadronic admissibility closure)**.** *On the admissible branch, the physical hadronic Hilbert space is*

$$\mathcal{H}_{\rm had} := \overline{\mathcal{A}_{\rm sing}|\Omega\rangle}.$$

All physical hadronic states lie in $\mathcal{H}_{\rm had}$, while color-nonsinglet free-word sectors are removed by $P_{\rm adm}$.

Proof. Center neutrality and seam-evenness are already the carrier-side admissibility rules for physical hadrons. Passing to the norm closure of the singlet admissible algebra acting on the vacuum gives the completed hadronic space, and no state with $z(w) \neq 1$ survives the selector. □

Theorem 10.31 (Anomaly cancellation and seam inflow)**.** *On the admissible branch,*

$$I_6^{\rm tot} = 0, \qquad \delta_\Lambda S_{\rm bulk} + \delta_\Lambda S_{\eta,\Sigma} = 0.$$

Proof sketch. The one-family packet S^+ is exactly the anomaly-free Standard Model family including ν^c. The remaining seam contribution is carried by the relative η term, so gauge variation cancels between bulk and seam sectors rather than being left as an uncancelled boundary defect. □

Lemma 10.32 (Topological sector positivity)**.** *On the admissible branch every topological sector weight satisfies*

$$Z_Q \geq 0.$$

Proof. Apply [TFPT cross-reference: `lem:retained-gamma5-hermiticity`]. The nonzero spectrum of each $\mathcal{D}_f$ occurs in paired sets $\pm i\lambda_n$, while the branch mass eigenvalues are strictly positive by [TFPT cross-reference: `thm:sheet-cp-protection`]. Therefore each sector determinant is nonnegative, and so is the full sector weight. □

Lemma 10.33 (Nontrivial topological support on the closed branch)**.** *On the closed branch the primitive topological sector has strictly positive weight:*

$$Z_1 = Z_{-1} > 0.$$

Proof. By [TFPT cross-reference: `thm:boundary-winding-control`], the canonical branch has unit seam winding, so the primitive topological class $Q = 1$ is nonempty on the admissible branch. Choose an admissible representative ϕ_1 in that class. By [TFPT cross-reference: `thm:sheet-cp-protection`], the same branch carries a genuine compact determinant angle with primitive winding number one, so ϕ_1 has finite admissible action and belongs to the closed branch sector $Q(\phi_1) = 1$. Let $\mathcal{U}_1$ be a sufficiently small open neighborhood of ϕ_1 inside that sector. The local Boltzmann density is strictly positive on $\mathcal{U}_1$, and Lemma 10.32 gives nonnegative sector weights. Therefore

$$Z_1 \geq \int_{\mathcal{U}_1} e^{-S_{\rm adm}[\phi]}\, d\mu(\phi) > 0.$$

The antiunitary sheet symmetry exchanges $Q = 1$ and $Q = -1$, hence $Z_{-1} = Z_1 > 0$. □

Theorem 10.34 (Unconditional strong-CP closure on the admissible branch)**.** *On the admissible branch one has*

$$\arg\det M_u = \arg\det M_d = 0, \qquad \bar{\theta} = 0.$$

Moreover the free energy

$$F(\theta) := -V^{-1} \log Z_{\rm adm}(\theta)$$

has its unique minimum at

$$\theta = 0 \mod 2\pi.$$

Weak CP may still survive through

$$V_{\mathrm{CKM}} = U_{u,L}^{\dagger} U_{d,L}.$$

Proof. [TFPT cross-reference: `thm:sheet-cp-protection`] gives

$$\arg\det M_u = \arg\det M_d = 0.$$

By Lemma 10.32, the admissible partition function admits a sector decomposition

$$Z_{\mathrm{adm}}(\theta) = \sum_{Q\in\mathbb{Z}} Z_Q e^{iQ\theta}, \qquad Z_Q \geq 0.$$

The antiunitary sheet symmetry of [TFPT cross-reference: `thm:sheet-cp-protection`] gives

$$\mathcal{C}_\Sigma \, Q \, \mathcal{C}_\Sigma^{-1} = -Q,$$

so the sector weights satisfy $Z_Q = Z_{-Q}$. Hence

$$\begin{aligned} Z_{\mathrm{adm}}(\theta) &= Z_0 + 2\sum_{Q>0} Z_Q \cos(Q\theta) \\ &\leq Z_0 + 2\sum_{Q>0} Z_Q = Z_{\mathrm{adm}}(0). \end{aligned}$$

By Lemma 10.33, one has $Z_1 > 0$. Therefore for every $\theta \not\equiv 0 \mod 2\pi$,

$$Z_{\mathrm{adm}}(\theta) \leq Z_{\mathrm{adm}}(0) - 2Z_1\big(1-\cos\theta\big) < Z_{\mathrm{adm}}(0),$$

which implies

$$F(\theta) > F(0) \qquad \text{for every } \theta \not\equiv 0 \mod 2\pi.$$

Thus the unique minimum is at $\theta = 0$, and therefore $\bar{\theta} = 0$. □

Theorem 10.35 (Joint discrete admissible sector from the boundary primitive kernel)**.** *For every admissible boundary primitive kernel $\mathfrak{T}^{\partial}_{\mathrm{ker}}$, the compatible discrete datum*

$$\mathfrak{D}_{\mathrm{disc}}(\mathfrak{T}^{\partial}_{\mathrm{ker}})$$

is a singleton. Its unique element is

$$d^{\star}_{\mathrm{disc}} := \Big(E_3 \oplus E_2,\, Y,\, S^+,\, X_f^{\circ},\, [\nabla_F^{\star}],\, c_1(L_2),\, c_1(L_3),\, N_{\mathrm{fam}},\, \theta_{\mathrm{eff}}\Big)$$

with

$$X_f^{\circ} \cong \mathbb{P}^1 \setminus \mu_4, \qquad N_{\mathrm{fam}} = 3, \qquad (g,b,s) = (5,3,2), \qquad \dim S^+ = 16,$$

$$S^+ = \Lambda^{\mathrm{even}}(E_3 \oplus E_2),$$

$$Y = \mathrm{diag}\big(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}\big),$$

$$(c_1(L_2), c_1(L_3)) = (1,0), \qquad \theta_{\mathrm{eff}} = 0.$$

Equivalently, carrier data, family data, determinant data, and the effective strong angle are fixed jointly from the same boundary primitive kernel.

Proof. We prove the discrete blocks in the only order that uses no downstream uniqueness claim.

Step 1: primitive winding. By [TFPT cross-reference: `thm:primitive-seam-generator`], the primitive seam class is the positive generator

$$[u_\Sigma] = 1 \in K^1(S^1) \cong \mathbb{Z},$$

and its associated spectral flow is

$$\mathrm{SF}(U_\Sigma) = 1.$$

Hence the primitive branch carries exactly one elementary seam winding. Write this winding number as

$$n := \mathrm{SF}(U_\Sigma) = 1.$$

Step 2: family geometry and family multiplicity. The lifted spin action on the oriented normal two-plane closes only after 4π, so one primitive winding is realized by four quarter-turn corners. Therefore the compact family quotient carries exactly four $\mathbb{Z}_4$ orbifold points, and removing them produces

$$X_f^\circ \cong \mathbb{CP}^1 \setminus \{q_1, q_2, q_3, q_4\}.$$

The lifted seam symmetry acts by the symmetries of a square, so the marked configuration admits a faithful D_4 action. By the classification of finite subgroups of $PGL_2(\mathbb{C})$, every faithful D_4 action on four marked points of $\mathbb{CP}^1$ is Möbius conjugate to the square configuration

$$\{1, -1, i, -i\}.$$

Hence

$$X_f^\circ \cong \mathbb{P}^1 \setminus \mu_4.$$

The corner deficit count is then

$$N_{\mathrm{fam}} := \sum_{j=1}^{4} \left(1 - \frac{1}{4}\right) = 4 \cdot \frac{3}{4} = 3.$$

Because $\chi(X_f^\circ) = 2 - 4 = -2$, one also has

$$H_1(X_f^\circ, \mathbb{Z}) \cong \mathbb{Z}^3, \qquad H^1(X_f^\circ, \mathbb{C}) \cong \mathbb{C}^3.$$

Thus the family multiplicity and the family geometry are fixed already at the discrete level.

Step 3: carrier closure. By [TFPT cross-reference: `cor:bosonic-rank-two`], the compact Higgs index has already fixed

$$\dim E_+ = 2.$$

By [TFPT cross-reference: `thm:yukawa-forces-32`], the primitive indecomposable generator of the actual branch Yukawa tensor then forces

$$\dim E_- = 3.$$

Hence

$$(b, s) = (3, 2),$$
$$g = b + s = 5,$$
$$\dim S^+ = 2^{g-1} = 16.$$

Derived consistency check. With $N_{\mathrm{fam}} = 3$ from Step 2 and $(b, s) = (3, 2)$ from the present step, one recovers

$$\Omega_{\mathrm{adm}} = N_{\mathrm{fam}} \dim S^+ = 3 \cdot 16 = 48 = (g-1)(b^2 + s^2 - 1) = 4 \cdot 12.$$

Step 4: hypercharge and one-family packet. With $(b, s) = (3, 2)$ fixed, the even half-spin packet is

$$S^+ = \Lambda^{\mathrm{even}}(E_3 \oplus E_2).$$

Primitive unimodularity on the two-eigenvalue carrier gives

$$3q_- + 2q_+ = 0.$$

The unique primitive integer solution with $q_- < 0 < q_+$ is

$$(q_-, q_+) = (-2, 3).$$

Hence

$$X = -2P_3 + 3P_2, \qquad Y = \frac{X}{6} = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right).$$

Step 5: determinant class and strong CP. The determinant-line data are classified by the clutching pair $(c_1(L_2), c_1(L_3))$. On the admissible branch the compact bosonic index satisfies

$$\mathrm{Ind}(\mathcal{B}^+_{E_2}) \geq 0, \qquad \mathrm{Ind}(\mathcal{B}^+_{E_3}) \geq 0.$$

Existence of a seam-even Higgs doublet forces $\mathrm{Ind}(\mathcal{B}^+_{E_2}) \geq 1$, while absence of an unavoidable light color triplet forces $\mathrm{Ind}(\mathcal{B}^+_{E_3}) = 0$. Minimal clutching therefore gives the unique pair

$$(c_1(L_2), c_1(L_3)) = (1, 0).$$

For the determinant-phase sector, reflection positivity and determinant-line conjugation symmetry make the admissible free energy even in θ_{eff}, and the positive-transfer theorem gives a unique minimum at the origin. Hence

$$\theta_{\mathrm{eff}} = 0.$$

All discrete blocks are therefore fixed jointly by the same boundary primitive kernel. Thus

$$\mathfrak{D}_{\mathrm{disc}}(\mathfrak{T}^{\partial}_{\mathrm{ker}}) = \{d^{\star}_{\mathrm{disc}}\}.$$

□

Corollary 10.36 (Derived post-carrier kernel)**.** *The tuple*

$$\mathfrak{T}_{\mathrm{ker}} := (\mathcal{A}, \mathcal{H}, D, J, \Gamma, \iota_C, P_{\mathrm{prim}}, P_{\mathrm{adm}}, E_3 \oplus E_2, Y, [u_\Sigma], c_3)$$

is a theorem-level derived object of $\mathfrak{T}^{\partial}_{\mathrm{ker}}$. In particular the old post-carrier kernel is no longer primitive; it is reconstructed from the unique discrete datum $d^{\star}_{\mathrm{disc}}$ together with the full selector $P_{\mathrm{adm}} = P_{\mathrm{prim}} P_{\mathrm{sing}} P_{\Theta}$.

Proof. By Theorem 10.35, the carrier split $E_3 \oplus E_2$, the charge operator Y, the determinant class, and the family geometry are already fixed by $\mathfrak{T}^{\partial}_{\mathrm{ker}}$. The projector P_{sing} is then determined by the fixed carrier, and P_{Θ} is determined by the fixed determinant class. Therefore

$$P_{\mathrm{adm}} = P_{\mathrm{prim}} P_{\mathrm{sing}} P_{\Theta}$$

is fully determined. The displayed tuple is thus derived, not primitive. □

Definition 10.37 (Projected modular free energy and stratified master functional)**.** Fix the unique joint discrete datum $d^\star_{\rm disc}$ of Theorem 10.35, and let $P_{\rm adm}$ be the derived full selector of Corollary 10.36. For

$$\xi = (\alpha, \chi_{\rm geo}, \delta_{\rm ph}, \rho_{\rm vac})$$

on the retained admissible branch, define

$$\mathcal{M}_{d^\star_{\rm disc}}(\xi) := -\log Z_{\rm rel}^{P_{\rm adm}}[\xi] + \Re \log \det_{\zeta, P_{\rm adm}} (D_\xi^2 + \mu^2) + \frac{1}{2}\eta_{\Sigma, P_{\rm adm}}(D_\xi) - \log\det(1 - U_{6,\xi}) + \Gamma_{\rm vac}(\rho_\xi),$$

where every term is evaluated on the retained admissible sector $\mathrm{Ran}(P_{\rm adm})$. For a general admissible spectral-bordism state

$$\mathfrak{s} = (\mathcal{B}, \xi), \qquad \mathcal{B} \in \mathbf{SBord}^{\rm adm},$$

define the lexicographic barrier

$$\mathbb{B}_{\rm lex}(\mathfrak{s}) := \begin{cases} 0, & \mathcal{B} \cong \mathcal{B}_{\rm min}, \\ +\infty, & \text{otherwise}, \end{cases}$$

and the stratified master functional

$$\mathbb{M}(\mathfrak{s}) := \mathbb{B}_{\rm lex}(\mathfrak{s}) + \mathcal{M}_{d^\star_{\rm disc}}(\xi).$$

Thus the continuous retained-branch functional $\mathcal{M}_{d^\star_{\rm disc}}$ is the restriction of one stratified functional to the unique minimal admissible stratum.

Theorem 10.38 (Stratified master functional and continuous closure)**.** *On the retained admissible branch, the projected modular free energy factorizes as*

$$\mathcal{M}_{d^\star_{\rm disc}}(\xi) = \Gamma_{U(1)}(\alpha) + \Gamma_{\rm grav}(\chi_{\rm geo}) + \Gamma_{\rm tr}(\delta_{\rm ph}) + \Gamma_{\rm vac}(\rho_{\rm vac}) + \text{const}.$$

Its admissible critical points are therefore exactly the solutions of

$$\partial_\alpha \Gamma_{U(1)}(\alpha) = 0, \qquad \partial_{\chi_{\rm geo}} \Gamma_{\rm grav}(\chi_{\rm geo}) = 0, \qquad \partial_\delta \Gamma_{\rm tr}(\delta_{\rm ph}) = 0, \qquad \delta_\rho \Gamma_{\rm vac}(\rho_{\rm vac}) = 0.$$

Equivalently, the continuous closure laws are

$$F_{U(1)}(\alpha) = 0, \qquad \mathcal{R}_{\rm grav} = 0, \qquad P_{\rm tr}(\delta_{\rm ph}) = 0, \qquad \delta_\rho \Gamma_{\rm vac}(\rho_{\rm vac}) = 0.$$

In particular the continuous closure equations are derived from one primitive variational object rather than imposed as independent sector primitives. Equivalently, the stationary points of the stratified master functional $\mathbb{M}$ are exactly the stationary points of $\mathcal{M}_{d^\star_{\rm disc}}$ on the minimal admissible stratum.

Proof. By construction,

$$\mathbb{M}(\mathfrak{s}) = +\infty$$

off the unitary-equivalence class of the minimal admissible bordism $\mathcal{B}_{\rm min}$. Hence every stationary point of $\mathbb{M}$ must lie on that minimal stratum, where the barrier vanishes and

$$\mathbb{M}(\mathfrak{s}) = \mathcal{M}_{d^\star_{\rm disc}}(\xi).$$

Now apply Corollary 10.36: all continuous terms are evaluated on one and the same retained sector $\mathrm{Ran}(P_{\rm adm})$. On that branch the family, determinant, transport, and vacuum projectors commute, and the physical Hilbert space splits into orthogonal direct summands corresponding to the electromagnetic, geometric, transport, and vacuum blocks. For a block-diagonal operator, the relative partition function factorizes, the zeta determinant factorizes, and the eta invariant is additive. Therefore the logarithms split $\mathcal{M}_{d^\star_{\rm disc}}$ into a sum of four sector functionals plus a constant.

The $U(1)$ block is exactly the already-defined electromagnetic effective action, so

$$\partial_\alpha \Gamma_{U(1)}(\alpha) = F_{U(1)}(\alpha).$$

The geometric block is the reduced spectral Einstein functional, hence

$$\partial_{\chi_{\rm geo}} \Gamma_{\rm grav}(\chi_{\rm geo}) = \mathcal{R}_{\rm grav}.$$

The transport block is the logarithmic resolvent determinant of the exact C_6 transfer denominator, so

$$\partial_\delta \Gamma_{\rm tr}(\delta_{\rm ph}) = P_{\rm tr}(\delta_{\rm ph}).$$

Finally the vacuum block is by definition the admissible vacuum effective action, so

$$\delta_\rho \Gamma_{\rm vac}(\rho_{\rm vac}) = 0$$

is exactly its Euler–Lagrange equation. Thus the four continuous closure equations are the Euler–Lagrange equations of one projected modular free energy. □

Definition 10.39 (Sector solution sets)**.** Define

$$Z_{\rm disc} := \mathfrak{D}_{\rm disc}(\mathfrak{T}^\partial_{\rm ker}), \qquad Z_{U(1)} := \{\alpha : F_{U(1)}(\alpha) = 0\}, \qquad Z_{\rm grav} := \{\chi_{\rm geo} : \mathcal{R}_{\rm grav}(\chi_{\rm geo}) = 0\},$$
$$Z_{\rm tr} := \{\delta_{\rm ph} : P_{\rm tr}(\delta_{\rm ph}) = 0\}, \qquad Z_{\rm vac} := \{\rho_{\rm vac} : \delta_\rho \Gamma_{\rm vac}(\rho_{\rm vac}) = 0\}.$$

Let

$$\mathcal{I}_0 := (P_{\rm adm}, \iota_C, [u_\Sigma], E_3 \oplus E_2, Y, c_3)$$

be the upstream invariant tuple fixed before the continuous closure step.

Theorem 10.40 (Intrinsic branch as a singleton fiber product)**.** *For every admissible boundary primitive kernel $\mathfrak{T}^\partial_{\rm ker}$, the intrinsic closed branch is the singleton fiber product*

$$\mathcal{Z}_{\rm cl} \cong Z_{\rm disc} \times_{\mathcal{I}_0} Z_{U(1)} \times_{\mathcal{I}_0} Z_{\rm grav} \times_{\mathcal{I}_0} Z_{\rm tr} \times_{\mathcal{I}_0} Z_{\rm vac},$$

and therefore

$$\mathcal{Z}_{\rm cl} = \{x^\star_{\rm cl}\}.$$

No Jacobian on mixed discrete and continuous variables is needed.

Proof. By Theorem 10.35,

$$Z_{\rm disc} = \{d^\star_{\rm disc}\}.$$

By the exact electromagnetic theorem,

$$Z_{U(1)} = \{\alpha_\star\}.$$

By coercivity and strict convexity of the reduced Einstein functional,

$$Z_{\rm grav} = \{\chi^\star_{\rm geo}\}.$$

By the algebraic transport theorem together with the word-length lock,

$$Z_{\rm tr} = \{\delta^\star_{\rm ph}\}.$$

By the positive-transfer, thermodynamic-limit, and gap-stability theorems,

$$Z_{\rm vac} = \{\rho^\star_{\rm vac}\}.$$

Each singleton carries its compatibility map to the same upstream invariant tuple $\mathcal{I}_0$. That common tuple is already fixed earlier: the discrete theorem fixes $E_3 \oplus E_2$ and Y, the derived selector theorem fixes $P_{\rm adm}$, and the seam theorem fixes $[u_\Sigma]$ and c_3. Hence all compatibility maps land at the same point of $\mathcal{I}_0$. The fiber product of singleton sets over one common point is again a singleton. Therefore

$$\mathcal{Z}_{\rm cl} = \{x^\star_{\rm cl}\}.$$

□

Theorem 10.41 (Internal analytic closure as a corollary of primitive completion)**.** *For every admissible one-sided boundary datum, the analytic closure blocks needed for rigid reconstruction hold internally on the unique intrinsic branch determined by Theorems 10.35, 10.38 and 10.40.*

Proof. The Calderón and APS block is already fixed upstream by the primitive boundary theorem chain. The family, determinant, and effective strong-angle blocks are fixed by the joint discrete theorem. The electromagnetic, geometric, transport, and vacuum blocks are fixed by the continuous master functional together with the already-proved positivity, gap, and thermodynamic-limit theorems. Because the intrinsic branch is a singleton by Theorem 10.40, all analytic closure blocks hold on one and the same branch, internally and without any extra closure package. □

Theorem 10.42 (Canonical rigid reconstruction and essential injectivity on the rigid image)**.** *Let $\mathfrak{T}^{\rm min}_\partial$ be an admissible one-sided boundary datum and let $\mathfrak{T}^\partial_{\rm ker}$ be its boundary primitive kernel. Then the rigid tuple*

$$\mathfrak{C}(\mathfrak{T}^\partial_{\rm ker}) := \left(E_3 \oplus E_2,\, Y,\, S^+,\, X^\circ_f,\, [\nabla^\star_F],\, \Phi^\star,\, \chi^\star_{\rm geo},\, \delta^\star_{\rm ph},\, \rho^\star_{\rm vac},\, \theta_{\rm eff} = 0,\, U^\star_\Sigma,\, \Lambda^\star_{\rm IR}\right)$$

is uniquely determined. In particular the reconstruction functor is essentially injective on objects on its rigid image, and every two rigid realizations coming from the same boundary datum are uniquely unitarily isomorphic.

Proof. The joint discrete theorem fixes $E_3 \oplus E_2$, Y, S^+, X°_f, $[\nabla^\star_F]$, the determinant class, and $\theta_{\rm eff} = 0$. The seam theorem fixes $[u_\Sigma]$ and c_3. The continuous master-functional theorem fixes $\alpha_\star$, $\chi^\star_{\rm geo}$, $\delta^\star_{\rm ph}$, and $\rho^\star_{\rm vac}$. The downstream definitions of $\Phi^\star$, $U^\star_\Sigma$, and $\Lambda^\star_{\rm IR}$ are then evaluated on that unique branch. Hence every displayed entry of $\mathfrak{C}(\mathfrak{T}^\partial_{\rm ker})$ is uniquely determined.

If two rigid realizations arise from the same boundary datum, then all displayed entries agree. The carrier, seam, family, determinant, transport, and vacuum blocks each admit only the unique unitary intertwiner preserving the already-fixed theorem-level structures, and these intertwiners are compatible because they preserve the same upstream invariant tuple

$$(P_{\rm adm}, \iota_C, [u_\Sigma], E_3 \oplus E_2, Y, c_3).$$

Their direct sum or tensor product therefore yields one global unitary intertwiner. Thus the rigid image is essentially injective on objects, with unique unitary isomorphism on that image. □

11 Source Extraction Map

Source extraction map

Use `../tfpt-42.tex`:

- Sections 3.3, 3.4, and 3.5 for carrier normal form, internal stabilizer, one-family packet, character identities, trace identities, and abelian index.
- Sections 5.1–5.3 only for carrier-relevant compression identities.
- Sections 6.1–6.6 for families, occupancy, and compact Higgs index.
- Sections 8.11–8.15 for the Yukawa-based discharge of carrier premises.
- Section 8.19 and Section 9.11 only for the joint discrete admissible sector.

Editorial guardrail

Delete or defer exact electromagnetic numerics, CKM/PMNS, masses, OS reconstruction, cosmology, and E8. The paper stands or falls on the carrier theorem and the representation audit.

Exported objects

Exports: $E_3 \oplus E_2$, Y, $S^+ = \Lambda^{\text{even}} E$, G_{phys}, $N_{\text{fam}} = 3$, $\Omega_{\text{adm}} = 48$, $N_\Phi = 1$, and $b_1 = 41/10$.

12 Not Used Here

Exact α numerics, CKM/PMNS closure, pole-mass ledgers, OS/CAR reconstruction, local Minkowski nets, scattering, gravity/metrology, cosmology, CMB targets, and E8 grammar are not used as proof inputs in this paper.