Admissibility, Strong CP, and Nonperturbative QFT Closure

on the TFPT Branch

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Abstract

This paper isolates the analytic closure layer of TFPT. The selector P_{adm} is treated as a physical admissible-sector construction, while the dynamics is carried by Z_{rel}, admissible Schwinger distributions, Osterwalder–Schrader reconstruction, the local Minkowski net, stable massive scattering, and the exact admissible RG flow.

Scope box: inputs, contribution, non-claims, audit surface

Inputs from previous papers. Paper 1 supplies P_{prim}. Paper 2 supplies the carrier and discrete determinant data needed for P_{sing} and P_{Θ}. Paper 3 is not logically required except for optional observable cross-references.
New theorem contribution. Conditional nonperturbative closure on the admissible sector:

$$P_{\mathrm{adm}} = P_{\mathrm{prim}} P_{\mathrm{sing}} P_{\Theta}, \qquad \theta_{\mathrm{eff}} = 0, \qquad \arg\det M_u = \arg\det M_d = 0,$$

together with reflection positivity, OS reconstruction, local Minkowski net, stable massive scattering, and exact admissible flow.
Not claimed here. No α detail calculation, no CMB, no E8, and no full empirical tables.
Falsification or audit surface. The paper fails if selector and dynamics are conflated, if positivity/gap hypotheses are hidden, or if strong-CP closure uses an inadmissible phase convention.

Editorial guardrail

Preferred claim language: *conditional nonperturbative closure under explicitly stated admissibility, positivity, and gap hypotheses*. Avoid unqualified “proved QFT” rhetoric.

Editorial guardrail

Every analytic theorem in this paper is to be read with its hypotheses in the statement, not as an unconditional construction of four-dimensional QFT. The standing checklist is: admissible Schwinger family, temperedness, Euclidean covariance, reflection positivity, clustering, locality-domain regularity, and a stated massive-sector gap where scattering is invoked.

Claim contract

Claim. P_{adm} selects the physical sector; strong CP closes; QFT reconstruction is conditional on stated analytic hypotheses.
Inputs. P_{prim}, carrier/determinant data, admissibility complex, positivity/gap hypotheses.
First assumptions. Reflection positivity, temperedness, Euclidean covariance, clustering, locality-domain regularity, sector gap.
Proof status. Conditional nonperturbative closure / standard-theorem interface.
Kill condition. Hidden positivity/gap assumption, nonzero strong-CP angle, or failure of OS/local-net hypotheses.

Contents

1 Selector Versus Dynamics

The central distinction is

$$P_{\mathrm{adm}} \quad \text{selects the physical sector,}$$

whereas dynamics is carried by

$$Z_{\mathrm{rel}} \Rightarrow \{S_n^T\} \Rightarrow (\mathcal{H}_{\mathrm{adm}}, \mathfrak{A}_{\mathrm{adm}}) \Rightarrow \Gamma_k \Rightarrow \Gamma_{\mathrm{TFPT}}^{\mathrm{ren}}.$$

This separation must be repeated in the introduction and conclusion because it is the main defense against overclaiming.

2 Full Admissibility Complex

After carrier and determinant data are fixed, the full selector is

$$P_{\mathrm{adm}} = P_{\mathrm{prim}} P_{\mathrm{sing}} P_{\Theta}.$$

The paper should retain the two-stage upgrade from P_{prim} to P_{adm}, the Hodge projector description, and the exact role of the gap projection. Carrier material is summarized only as an input.

3 Strong CP Closure

The strong-CP sector is stated as an admissibility result:

$$\theta_{\mathrm{eff}} = 0, \qquad \arg\det M_u = 0, \qquad \arg\det M_d = 0.$$

The argument should connect hadronic singlet selection, determinant structure, γ_5-Hermiticity, and the sheet involution without importing phenomenological tuning.

4 Reflection Positivity and OS Reconstruction

The reflection-positivity block should state all assumptions on the admissible Schwinger functions, fermionic positivity, quotient form, and domain regularity. The OS theorem is used as a standard interface only after the admissible-sector positivity hypotheses are named.

5 Local Net and Scattering

The local Minkowski net, Haag–Ruelle/LSZ massive scattering, and dressed massless interface are treated as conditional closure outputs. The paper must distinguish stable massive scattering from low-curvature dressed massless behavior.

6 Exact Admissible Flow

The exact admissible RG flow is the analytic continuation of the same sector:

$$\partial_k \Gamma_k = \frac{1}{2} \operatorname{STr} \left[(\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k \right]_{\mathrm{adm}},$$

with the admissible projection included as part of the flow definition. The observable hierarchy is included only as far as it supports the QFT closure layer.

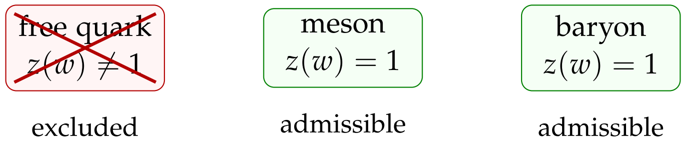


Figure 1. Hadronic admissibility as singlet selection.

7 Main Technical Development

This section contains the main technical development assigned to this paper by the TFPT 4.5 clean split. Cross-paper background is referenced through dependency and scope boxes; extended backend material is kept in the Technical Companion.

7.1 Hadronic admissibility as singlet selection

Proposition 7.1 (Admissible color words)**.** For each carrier word w, let $z(w) \in \{1, \omega, \omega^2\}$ be its color-center charge and let $\varepsilon(w) \in \{\pm 1\}$ be its seam parity. Physical hadronic words satisfy

$$z(w) = 1, \qquad \varepsilon(w) = +1.$$

This is a superselection statement on the retained branch, not a derivation of dynamical confinement from the microscopic Yang–Mills flow. The corresponding hadronic Hilbert space will be written explicitly in [TFPT cross-reference: `sec:closure-theorems`]. A schematic version is shown in Figure 1.

The transport-sector QCD string tension is written as

$$\sigma_{\mathrm{QCD}} = c_3^2 \chi_{\mathrm{QCD}}^2,$$

so the relative Wilson loop obeys

$$\langle W(C) \rangle_{\mathrm{rel}} \sim e^{-\sigma_{\mathrm{QCD}} A(C)}.$$

Throughout the manuscript we write σ_{QCD} explicitly for the QCD string tension to keep it lexically distinct from the local Weyl conformal factor σ of the geometric branch and from the standard-deviation σ used in benchmark tables. The sector-dependent scale χ_{QCD} is the minimal notation needed to avoid confusing the hadronic transport channel with the gravitational $\chi_{\mathrm{geo},0}$ background; the two channels are linked by the common underlying primitive response χ_{seed} but not by a naive equality of scales.

7.2 Transport-side ingredients for the strong-CP sector

Define

$$\bar{\theta} = \theta_{\mathrm{QCD}} + \arg\det M_u + \arg\det M_d. \tag{1}$$

Definition 7.2 (Sheet-CP symmetry on the admissible branch)**.** Define the admissible sheet-CP operator by

$$\mathcal{C}_\Sigma := CP \circ \tau_{\mathrm{dbl}}.$$

Theorem 7.3 (Determinant-line phase and strong-CP seed on the admissible branch)**.** *On the closed branch the hard holonomy factorization gives, for $f = u, d$,*

$$Y_f = D_{L,f} U_f^\star D_{R,f}, \qquad D_{L,f}, D_{R,f} > 0, \qquad U_f^\star \in SU(3)_F.$$

For

$$M_f = \frac{v_{\mathrm{geo}}}{\sqrt{2}} Y_f$$

one has

$$\det M_f > 0, \qquad \arg\det M_u = \arg\det M_d = 0.$$

Moreover the polar unitary factor

$$U_f^{\mathrm{pol}} := M_f(M_f^\dagger M_f)^{-1/2}$$

lies in $SU(3)_F$. The antiunitary sheet operation preserves the admissible relative action Γ_{rel}:

$$\mathcal{C}_\Sigma\,\Gamma_{\mathrm{rel}}\,\mathcal{C}_\Sigma^{-1} = \Gamma_{\mathrm{rel}}, \qquad \mathcal{C}_\Sigma\,Q\,\mathcal{C}_\Sigma^{-1} = -Q,$$

so the sheet operation acts as a stability symmetry of the same branch. The same admissible seam operator determines the canonical determinant-line phase

$$a_\Sigma := \arg\det_{\mathrm{adm}} U_\Sigma \in S^1_{\det},$$

whose canonical winding is inherited from the unit boundary winding of the seam class. Weak CP may still survive through

$$V_{\mathrm{CKM}} = U_{u,L}^\dagger U_{d,L}.$$

Proof. From the hard holonomy factorization one gets

$$\det M_f = \left(\frac{v_{\mathrm{geo}}}{\sqrt{2}}\right)^3 \det D_{L,f}\det D_{R,f}\det U_f^\star > 0,$$

because the diagonal factors are positive and $\det U_f^\star = 1$. Hence

$$\arg\det M_f = 0.$$

Equivalently, in any admissible biunitary polar form

$$M_f = U_{f,L}H_f U_{f,R}^\dagger, \qquad H_f > 0, \qquad U_{f,L}, U_{f,R} \in SU(3)_F,$$

the spectral theorem gives $\det H_f > 0$ and therefore

$$\det M_f = \det(U_{f,L})\det(H_f)\overline{\det(U_{f,R})} = \det H_f > 0,$$

so the same determinant-phase suppression follows directly from $\det H_f > 0$ and $\det U_{f,L} = \det U_{f,R} = 1$. For the polar unitary factor one has

$$\det U_f^{\mathrm{pol}} = \frac{\det M_f}{|\det M_f|} = 1,$$

so $U_f^{\mathrm{pol}} \in SU(3)_F$. Independently, the deck involution τ_{dbl} implements sheet exchange on the admissible branch, so composing it with CP yields an exact antiunitary symmetry of the relative action while flipping the sign of the topological charge. This shows that the sheet operation stabilizes the branch already selected by the determinant structure, rather than generating the vanishing from scratch. The same seam operator already carries a canonical determinant-line phase, and unit seam spectral flow upgrades that phase to the natural compact strong-CP angle variable of the branch. Weak CP may still survive through a nontrivial Jarlskog invariant in the relative family holonomies. □

Corollary 7.4 (RG and matching invariance of determinant-phase suppression)**.** *In every renormalization scheme that respects* C_Σ*,* P_{adm}*, and the carrier structure, the determinant-phase suppression*

$$\arg\det M_u = \arg\det M_d = 0$$

of [TFPT cross-reference: `thm:sheet-cp-protection`] *is preserved under the declared RG flow and matching map. The same determinant-line phase* a_Σ *remains scheme-stable, so the structural input to Theorem 8.22 survives RG flow as well.*

Lemma 7.5 (Retained-branch γ_5-Hermiticity)**.** *For each quark sector* $f = u, d$*, let* $D_{E,f}$ *be the Euclidean kinetic Dirac operator on the admissible branch and define*

$$\mathcal{D}_f := D_{E,f} + M_f P_R + M_f^\dagger P_L, \qquad P_{L,R} := \frac{1 \mp \gamma_5}{2}.$$

Then

$$\mathcal{D}_f^\dagger = \gamma_5 \mathcal{D}_f \gamma_5.$$

Proof. The Euclidean kinetic operator is anti-Hermitian and anticommutes with γ_5. The mass block on the admissible branch is the positive polar mass map of [TFPT cross-reference: `thm:sheet-cp-protection`], hence enters as the Hermitian chiral pair $M_f P_R + M_f^\dagger P_L$. The displayed identity follows immediately. □

The positive-branch clause is still essential here: the closure statement is about the physical mass map once the admissible transport branch has been fixed, not about an arbitrary unitary dressing of the same singular values. What changes in the present rewrite is the logical source of the null result: the branch no longer relies on a promise not to insert a bad counterterm by hand, but on the polar structure first, the determinant line second, and the antiunitary sheet symmetry third. The later strong-CP theorem closes the effective angle by combining exactly these three ingredients with vacuum positivity.

Remark (Admissibility as a unified three-sector motif)*.* The admissibility pair $(\mathsf{Adm}, P_{\mathrm{adm}})$ acts as a single structural principle with three sector projections:

- **Electroweak:** seam-even selection picks the unique light Higgs doublet ($\tau_{\mathrm{dbl}}^* \Phi = +\Phi$).
- **QCD:** center neutrality selects confined hadronic words ($z(w) = 1$, $\varepsilon(w) = +1$).
- **Strong CP:** the determinant structure, γ_5-Hermiticity, and the sheet involution close the strong angle internally on the admissible branch.

These are not three independent rules bolted onto different sectors. They are three projections of the same admissible-versus-inadmissible boundary: seam parity, center charge, and sheet exchange are all encoded in the double-cover and involution data of the primitive core. This unification is the conceptual backbone of the transport program.

8 Closure theorems from the admissibility complex

The preceding two sections introduced the carrier-compatible geometric and transport architecture. The primitive admissibility projector P_{prim} was already placed in the primitive section so that the proof direction runs from the boundary datum through the derived closed datum to P_{prim}, and only then to the later sector theorems. What remains in this section is therefore (i) the upgrade of P_{prim} to the full admissibility projector P_{adm} once the discrete carrier and determinant data have been fixed, and (ii) the closure stack that follows from that full projector together with the family, Higgs, positivity, and reflection-stability inputs collected first.

Definition 8.1 (Full admissibility complex)**.** Once the discrete carrier and determinant data are fixed on the admissible branch, the Hilbert space carries a color factor and a determinant involution. Write

$$\mathcal{H} = \mathcal{H}_\Sigma \otimes \mathcal{H}_c \otimes \mathcal{H}_\Theta \otimes \mathcal{H}_t, \qquad \mathcal{H}_{\rm ext} := \mathcal{H} \otimes \Lambda(\eta_\Sigma, \eta_c, \eta_\Theta, \eta_g),$$

with carrier-side color generator Z_c and determinant-sector involution Q_Θ satisfying

$$\iota_C^2 = \mathbf{1}, \qquad Q_\Theta^2 = \mathbf{1}, \qquad Q_\Theta^\dagger = Q_\Theta.$$

Define the full structural selector

$$P_{\rm str} := P_{\Sigma,+} P_{\rm sing} P_\Theta, \qquad P_{\Sigma,+} := \frac{\mathbf{1} + \iota_C}{2}, \qquad P_\Theta := \frac{1}{2}(\mathbf{1} + Q_\Theta),$$

$$P_{\rm sing} := \frac{1}{3}(\mathbf{1} + Z_c + Z_c^2).$$

Let $\epsilon_{\rm adm} > 0$ and let $D_{\rm adm}$ be the admissible sector operator on $\mathrm{Ran}(P_{\rm str})$ commuting with ι_C, $P_{\rm sing}$, and Q_Θ. Define the elementary penalties

$$K_\Sigma := \sqrt{\lambda_\Sigma}\, \frac{\mathbf{1} - \iota_C}{2}, \qquad K_c := \sqrt{\lambda_c}\, \big(\mathbf{1} - P_{\rm sing}\big),$$

$$K_\Theta := \sqrt{\lambda_\Theta}\, \frac{\mathbf{1} - Q_\Theta}{2}, \qquad K_g := \sqrt{\lambda_{\rm gap}}\, \mathbf{1}_{[0,\epsilon_{\rm adm}^2/4)}\big(P_{\rm str} D_{\rm adm}^\dagger D_{\rm adm} P_{\rm str}\big).$$

The full admissibility differential is

$$Q_{\rm adm} := \eta_\Sigma^\dagger K_\Sigma + \eta_c^\dagger K_c + \eta_\Theta^\dagger K_\Theta + \eta_g^\dagger K_g,$$

and the full admissibility Laplacian is

$$\Delta_{\rm adm} := \{Q_{\rm adm}, Q_{\rm adm}^\dagger\}.$$

Theorem 8.2 (Two-stage upgrade from $P_{\rm prim}$ to $P_{\rm adm}$)**.** *On the closed branch obtained after the carrier theorem and the compact bosonic index closure,*

$$Q_{\rm adm}^2 = 0, \qquad \Delta_{\rm adm} = K_\Sigma^2 + K_c^2 + K_\Theta^2 + K_g^2 = K_{\rm adm},$$

and the full Hodge projector

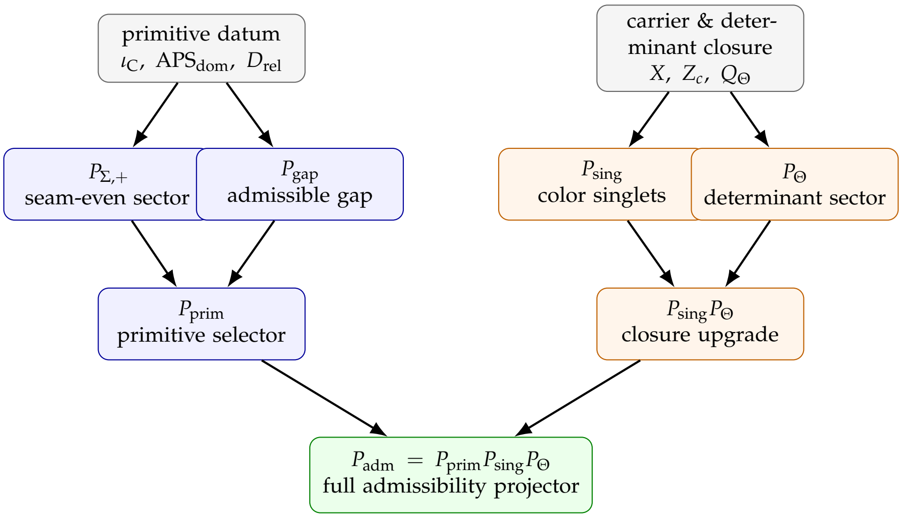
$$P_{\rm adm} := \Pi_{\ker \Delta_{\rm adm}} = \operatorname*{s-lim}_{\beta\to\infty} e^{-\beta K_{\rm adm}}$$

admits the two-stage factorization

$$P_{\rm adm} = P_{\rm prim}\, P_{\rm sing}\, P_\Theta = P_{\rm str}\, P_{\rm gap}, \qquad P_{\rm gap} := \mathbf{1}_{[\epsilon_{\rm adm}^2/4,\infty)}\big(P_{\rm str} D_{\rm adm}^\dagger D_{\rm adm} P_{\rm str}\big).$$

The first factor $P_{\rm prim}$ depends only on the primitive datum $(\iota_C, \mathrm{APS}_{\rm dom}, D_{\rm rel})$; the second factor $P_{\rm sing}P_\Theta$ depends only on the carrier and determinant data made available by the carrier and bosonic index theorems.

Proof sketch. The auxiliary generators $\eta_\Sigma^\dagger, \eta_c^\dagger, \eta_\Theta^\dagger, \eta_g^\dagger$ anticommute, so $Q_{\rm adm}^2$ reduces to the commutators of $K_\Sigma, K_c, K_\Theta, K_g$. The four commuting sectors $\mathcal{H}_\Sigma, \mathcal{H}_c, \mathcal{H}_\Theta, \mathcal{H}_t$ are acted on independently by these penalties, hence the mixed terms vanish and $\Delta_{\rm adm} = K_{\rm adm}$. The strong-coupling limit therefore projects onto the harmonic admissible sector. Because K_Σ and K_g already define $P_{\rm prim}$ on $\mathcal{H}_{\rm prim}$ and the additional penalties K_c, K_Θ commute with both, the full projector factorizes through $P_{\rm prim}$ multiplicatively, giving the two-stage form. □



Primitive data determine P_{prim}; carrier and determinant closure upgrade it to P_{adm}.

Figure 2. Two-stage upgrade from the primitive selector to the full admissibility projector. The left lane is fixed by the primitive datum, while the right lane enters only after carrier and determinant closure.

Corollary 8.3 (Factorized selector as a derived form)**.** *On the admissible branch the familiar factorized selector*

$$P_{\mathrm{adm}} = P_{\Sigma,+}P_{\mathrm{sing}}P_\Theta\, P_{\mathrm{gap}}$$

is recovered as a corollary of [TFPT cross-reference: `thm:padm-upgrade`]*; nothing in the primitive section depends on it.*

The two-stage selector geometry is summarized in Figure 2.

Theorem 8.4 (Calderón upgrade of the APS boundary domain)**.** *Let $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ be the provisional APS realization of the geometric anchor used in* [TFPT cross-reference: `thm:well-posed-primitive-dynamics`], *and let D_b be the canonical doubling of D_{rel} across Σ. Let $C(D_b)$ denote the Calderón projector of D_b on the seam collar. Then there is a canonical boundary-elliptic upgrade*

$$\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}} \xrightarrow{C(D_b)} \mathrm{APS}_{\mathrm{dom}}$$

such that:

(i) $\mathrm{APS}_{\mathrm{dom}}$ *lies in the same admissible elliptic class as* $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ *and preserves the self-adjoint Fredholm realization of the geometric anchor;*

(ii) the relative graph domain of D_{rel} is unchanged, so all primitive results that depended only on $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ *continue to hold with* $\mathrm{APS}_{\mathrm{dom}}$*;*

(iii) the family-pairing data $\mathrm{APS}_{\mathrm{fam}}$ *are induced canonically on the doubled admissible branch and are not a separate primitive datum.*

Proof sketch. The Calderón projector $C(D_b)$ is canonically associated with the doubled operator on the seam collar and produces a boundary-elliptic projection compatible with the provisional APS choice. Both $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ and $\mathrm{APS}_{\mathrm{dom}}$ are admissible elliptic boundary conditions for the same Dirac-type operator and therefore define equivalent self-adjoint Fredholm realizations of the geometric anchor (boundary-elliptic equivalence preserves the Sobolev graph domain). Consequently every statement made on the primitive scaffold under $\mathrm{APS}^{\mathrm{prov}}_{\mathrm{dom}}$ persists on the closed branch under $\mathrm{APS}_{\mathrm{dom}}$. The induction of $\mathrm{APS}_{\mathrm{fam}}$ from canonical doubling is a standard consequence of the cobordism structure of the doubled APS pair. □

Proposition 8.5 (Master closure roadmap from canonical doubling)**.** This proposition is a roadmap statement: it lists the closure conclusions that the individual theorems of this section establish from $\mathfrak{T}_\partial^{\min}$, the derived closed datum, the carrier theorem, and the canonical doubling. It is not used as an input to those proofs and is recovered as a master corollary at the end of the section. Let X_b be the canonical double across Σ and write

$$D_b = \begin{pmatrix} 0 & D_{\mathrm{rel}}^- \\ D_{\mathrm{rel}}^+ & 0 \end{pmatrix}, \qquad Q_{\mathrm{tot}} := Q_{\mathrm{adm}} + Q_{\mathrm{geo}}, \qquad \Delta_{\mathrm{tot}} := \{Q_{\mathrm{tot}}, Q_{\mathrm{tot}}^\dagger\}.$$

Then on the closed branch all of the following are derived from the same datum (each item is the output of a downstream theorem in this section, not an extra hypothesis here):

(1) the Calderón projector of D_b canonically upgrades the provisional APS choice $\mathrm{APS}_{\mathrm{dom}}^{\mathrm{prov}}$ to $\mathrm{APS}_{\mathrm{dom}}$ (cf. [TFPT cross-reference: `thm:apsdom-upgrade`]);

(2) the family pairing data are induced canonically, so no separate $\mathrm{APS}_{\mathrm{fam}}$ datum remains;

(3) the upgrade theorem [TFPT cross-reference: `thm:padm-upgrade`] produces

$$P_{\mathrm{adm}} := \Pi_{\ker \Delta_{\mathrm{adm}}} = P_{\mathrm{prim}}\, P_{\mathrm{sing}}\, P_\Theta,$$

and the doubled total closure package preserves $\mathrm{Ran}(P_{\mathrm{adm}})$;

(4)

$$\chi(X_f^\circ) = -2, \qquad F \simeq H^1(X_f^\circ, \mathbb{C}) \simeq \mathbb{C}^3, \qquad N_{\mathrm{fam}} = 3;$$

(5) the harmonic determinant line is parallel and therefore

$$\mathrm{Hol}(\nabla_F) \subset SU(3)_F;$$

(6) admissible bosonic, fermionic, and geometric reflection positivity hold on $\mathrm{Ran}(P_{\mathrm{adm}})$;

(7) the physical geometric sector takes the form

$$\mathcal{H}_{\mathrm{phys}} \cong \mathcal{H}_2 \otimes \mathcal{H}_{\mathrm{sc}} \otimes \mathcal{H}_{\mathrm{matt}},$$

with the scalaron carried by the positive $(\chi_{\mathrm{geo}}, R^2)$ block;

(8) $P_{\mathrm{adm}} \mathcal{H}_{\mathrm{rel}} P_{\mathrm{adm}}$ is lower bounded and coercive.

Definition 8.6 (Relative generating functional)**.** The relative Euclidean generating functional is normalized as

$$Z_{\mathrm{rel}}[J, \eta, \bar\eta] := \frac{Z_{\mathrm{adm}}[J, \eta, \bar\eta]}{Z_{\mathrm{ref}}[0,0,0]}.$$

The reference sector is therefore a fixed normalizing denominator rather than a second fluctuating measure that would mix with admissible positivity.

Lemma 8.7 (Admissible reflection positivity)**.** *Assume*

$$[\Theta, P_\Theta] = 0, \qquad \Theta P_{\mathrm{adm}} = P_{\mathrm{adm}} \Theta,$$

the admissible bosonic kernel satisfies the Bernstein-positivity remark above and is reflection-stable on the admissible sector, and the admissible fermionic CAR kernel is reflection positive on that same sector. Then every admissible observable O supported in $\tau > 0$ satisfies

$$\langle \Theta O, O \rangle_{\mathrm{rel}} = \frac{1}{Z_{\mathrm{ref}}[0,0,0]} \langle \Theta O, O \rangle_{\mathrm{adm}} \geq 0.$$

Proof. By definition,

$$\langle \Theta O, O \rangle_{\mathrm{rel}} = \frac{1}{Z_{\mathrm{ref}}[0,0,0]} \langle \Theta O, O \rangle_{\mathrm{adm}}.$$

The denominator is a fixed positive normalizing constant, so the sign is determined entirely by the admissible numerator.

Split O into its bosonic and fermionic parts on the retained admissible sector. For the bosonic part, the Bernstein representation of the admissible kernel writes the Euclidean covariance as a positive superposition of reflection-stable heat kernels. Reflection across $\tau = 0$ therefore sends every positive-time bosonic observable to a positive quadratic form, so the bosonic contribution to

$$\langle \Theta O, O \rangle_{\mathrm{adm}}$$

is nonnegative.

For the fermionic part, the CAR reflection-positivity hypothesis on the admissible subspace implies that the reflected two-point kernel is positive in the standard fermionic sense; equivalently, the corresponding Pfaffian / determinant form on positive-time test spinors is nonnegative. Because the bosonic and fermionic admissible factors commute inside the normalized expectation, their product remains nonnegative. Dividing by the fixed reference normalization preserves that sign. Hence

$$\langle \Theta O, O \rangle_{\mathrm{rel}} \geq 0.$$

□

Theorem 8.8 (Operative admissibility selector)**.** *Under* [TFPT cross-reference: `def:full-admissibility-comple` *the operator* P_{adm} *is idempotent on the retained branch and realizes the admissibility predicate on all sectors used by the closed output statements:*

$$\mathsf{Adm}(X) = 1 \iff P_{\mathrm{adm}} X = X.$$

Proof sketch. By [TFPT cross-reference: `thm:padm-upgrade`], $P_{\mathrm{adm}} = P_{\mathrm{prim}} P_{\mathrm{sing}} P_{\Theta}$ is the orthogonal projector onto the harmonic admissible sector of the full complex and is therefore idempotent by construction. [TFPT cross-reference: `cor:factorized-admissibility-selector`] recovers the explicit factorized selector. The predicate and operator languages agree once the positive transport sector is fixed. □

Theorem 8.9 (Topological family closure on the admissible branch)**.** *By* [TFPT cross-reference: `thm:master-closure,thm:topological-family-closure,cor:boundary-bulk-occupancy`],

$$\chi(X_f^\circ) = -2, \qquad F \cong \mathbb{C}^3, \qquad N_{\mathrm{fam}} = 3.$$

Consequently

$$\Omega_{\mathrm{adm}} = N_{\mathrm{fam}} \dim S^+ = 3 \times 16 = 48, \qquad \delta_{\mathrm{top}} = \Omega_{\mathrm{adm}}\, c_3^4.$$

Proof sketch. This theorem is now a direct consequence of the master closure theorem from Section 6. The topological family closure and the orbifold corner count have already been compressed there into one statement, so the present layer merely records the corresponding admissibility consequence. □

Theorem 8.10 (Bosonic index and local spectral-scale closure)**.** *On the admissible branch,*

$$\mathrm{Ind}_{\mathrm{rel}}^{P_{\mathrm{adm}}}(\mathcal{B}_{E_2}^+) = 1, \qquad \mathrm{Ind}_{\mathrm{rel}}^{P_{\mathrm{adm}}}(\mathcal{B}_{E_3}^+) = 0,$$

so the unique light seam-even bosonic zero mode is

$$\Phi \in \Gamma(E_2) \cong (1,2)_{1/2}, \qquad \tau_{\rm dbl}^{*}\Phi = +\Phi.$$

Moreover the same closure layer yields

$$\begin{aligned} \frac{\bar{M}_{\rm Pl}^2}{\lambda_\Sigma^2} &= \frac{\rho_\star}{2\pi^2}, & G_N\lambda_\Sigma^2 &= \frac{\pi}{4\rho_\star}, \\ \frac{v_{\rm geo}}{\bar{M}_{\rm Pl}} &= g_{\rm car}\,\beta_{\rm rad}^2 \exp\left[-\frac{\alpha^{-1}(0)+\delta_{\rm ph}}{5}\right], \\ G_N v_{\rm geo}^2 &= \frac{1}{8\pi} g_{\rm car}^2 \beta_{\rm rad}^4 \exp\left[-\frac{2(\alpha^{-1}(0)+\delta_{\rm ph})}{5}\right]. \end{aligned}$$

Here $v_{\rm geo}$ is the seam-even UV source scale. The physical electroweak benchmark is the residue matched quantity $v_{\rm phys}$ of [TFPT cross-reference: `thm:ew-geometric-to-physical-matching`].

Proof sketch. The bosonic sector is treated as a genuine Callias / Dirac / Dolbeault-type index problem rather than a scalar counting exercise. Relative heat-kernel closure together with [TFPT cross-reference: `prop:canonical-bosonic-normalization,lem:two-sheet-zero-mode-normalization,thm:a` produces the boundary-normalized Einstein branch, the canonical Higgs normalization, and the seam-even zero-mode formula for $v_{\rm geo}$. □

Definition 8.11 (Primitive Yukawa generator)**.** Let

$$\mathcal{I}_Y := \bigoplus_{p,q,r,t} \mathrm{Hom}_{G_{\rm car}}\Big((\Lambda^p E_- \otimes \Lambda^q E_+) \otimes (\Lambda^r E_- \otimes \Lambda^t E_+) \otimes E_+, \mathbb{C}\Big)$$

be the graded algebra of carrier-invariant Yukawa trilinears. A homogeneous nonzero invariant is called primitive if it is indecomposable, equivalently if it does not lie in the ideal generated by positive-degree invariants of strictly lower carrier degree.

Lemma 8.12 (Lowest primitive Yukawa generator on the closed branch)**.** *Assume* $\dim E_+ = 2$ *and let $\mathbb{Y}_{\rm br}$ be the branch Yukawa tensor of* [TFPT cross-reference: `thm:exact-transport-closure,thm:rigid-family` *Then the primitive generator of $\mathcal{I}_Y$ with two nontrivial fermionic legs is unique up to exchange of the two fermionic legs and belongs to*

$$\mathrm{Hom}_{G_{\rm car}}\big((E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+, \mathbb{C}\big).$$

Proof. The positive-block determinant neutrality gives $q+t+1=2$, hence $q+t=1$. Up to exchanging the two fermionic legs, $(q,t)=(1,0)$. Evenness of both fermionic legs then forces p odd and r even. If $p \geq 3$, the corresponding invariant factors through an extra negative spectator wedge and therefore lies in the ideal generated by lower carrier degree. Likewise, if $r \geq 4$, the invariant factors through a spectator bivector insertion and is again decomposable. Primitive indecomposability therefore forces $p=1$ and $r=2$. □

Theorem 8.13 (Carrier rigidity from the primitive component of the branch Yukawa tensor)**.** *Let*

$$E = E_- \oplus E_+$$

be the two-factor carrier and let

$$G_{\rm car} := S\big(U(E_-) \times U(E_+)\big), \qquad S^+ = \Lambda^{\rm even} E.$$

Let $\mathbb{Y}_{\mathrm{br}}$ *denote the branch Yukawa tensor induced by the canonical transport kernel and rigid family holonomy of* [TFPT cross-reference: `thm:exact-transport-closure,thm:rigid-family-local-system`]. *Decompose its carrier-homogeneous components as trilinear maps of the form*

$$B_{p,q;r,t} : (\Lambda^p E_- \otimes \Lambda^q E_+) \otimes (\Lambda^r E_- \otimes \Lambda^t E_+) \otimes E_+ \longrightarrow \mathbb{C}.$$

Let the primitive component of $\mathbb{Y}_{\mathrm{br}}$ *be the unique indecomposable generator of Definition 8.11 and lemma 8.12 with two nontrivial fermionic legs. Then, up to exchanging the two fermionic legs, that primitive component is necessarily of type*

$$(E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+ \longrightarrow \Lambda^3 E_- \otimes \Lambda^2 E_+ \cong \mathbb{C},$$

and therefore

$$\dim E_+ = 2, \qquad \dim E_- = 3.$$

Consequently

$$\dim \Lambda^2 E_+ = 1, \qquad \Lambda^2 E_- \cong E_-^\vee, \qquad \dim \Lambda^{\mathrm{even}} E = 16.$$

Hence the rigid carrier is

$$E = E_3 \oplus E_2, \qquad X = -2P_3 + 3P_2, \qquad Y = -\frac{1}{3}P_3 + \frac{1}{2}P_2.$$

Proof. By [TFPT cross-reference: `cor:bosonic-rank-two`], the compact Higgs index has already fixed

$$\dim E_+ = 2.$$

Write the primitive nonzero Yukawa component with two nontrivial fermionic legs as

$$B_{p,q;r,t} : (\Lambda^p E_- \otimes \Lambda^q E_+) \otimes (\Lambda^r E_- \otimes \Lambda^t E_+) \otimes E_+ \longrightarrow \mathbb{C}$$

and note that Lemma 8.12 forces primitiveness in the invariant-ring sense to have

$$(q,t) = (1,0), \qquad (p,r) = (1,2)$$

up to exchanging the two fermionic legs. Thus the primitive carrier component is exactly

$$(E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+.$$

Its negative factor contracts canonically as

$$E_- \otimes \Lambda^2 E_- \longrightarrow \Lambda^3 E_-,$$

so scalarity forces $\Lambda^3 E_-$ to be the top exterior power. Hence

$$\dim E_- = 3.$$

With

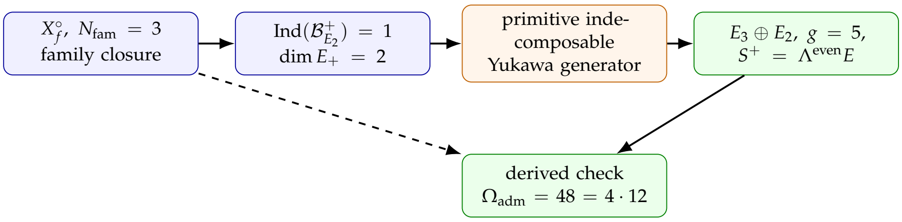
$$(\dim E_-, \dim E_+) = (3,2),$$

one obtains the carrier-signature consequences

$$\Lambda^2 E_- \cong E_-^\vee, \qquad \dim \Lambda^2 E_+ = 1,$$

and therefore

$$\dim \Lambda^{\mathrm{even}}(E_3 \oplus E_2) = 2^{5-1} = 16.$$



Carrier closure runs through Higgs rank and branch Yukawa rigidity; the old occupancy formula is only a downstream consistency identity.

Figure 3. Carrier closure after the proof-tree cut. The retained branch first fixes $\dim E_+ = 2$ through the compact Higgs index, then $\dim E_- = 3$ through the primitive indecomposable generator of the actual branch Yukawa tensor; only afterwards is $\Omega_{\mathrm{adm}} = 48 = 4 \cdot 12$ read as a consistency check.

Finally the unimodular primitive two-point generator is fixed by

$$3q_- + 2q_+ = 0, \qquad q_- < 0 < q_+, \qquad \gcd(q_-, q_+) = 1,$$

so $(q_-, q_+) = (-2, 3)$ and hence

$$X = -2P_3 + 3P_2, \qquad Y = \frac{X}{6} = -\frac{1}{3}P_3 + \frac{1}{2}P_2.$$

□

Lemma 8.14 (Uniqueness of the cubic weak invariant). *Let $E = E_- \oplus E_+$ be a two-factor carrier on the retained branch and let*

$$\Phi \in E_+$$

be the unique seam-even light boson selected by the compact bosonic index. Then

$$\dim \mathrm{Hom}_{G_{\mathrm{car}}}\big((E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+, \mathbb{C}\big) = 1$$

if and only if

$$(\dim E_-, \dim E_+) = (3, 2).$$

In particular, on the retained branch the up-type cubic invariant is not an extra choice but the unique admissible cubic invariant.

Proof. By Schur functor decomposition,

$$\mathrm{Hom}_{G_{\mathrm{car}}}\big((E_- \otimes E_+) \otimes \Lambda^2 E_- \otimes E_+, \mathbb{C}\big) \cong \mathrm{Hom}_{SU(E_-)}(E_- \otimes \Lambda^2 E_-, \mathbb{C}) \otimes \mathrm{Hom}_{SU(E_+)}(E_+ \otimes E_+, \mathbb{C}).$$

The first factor is nonzero iff $\Lambda^2 E_- \cong E_-^\vee$, hence iff $\dim E_- = 3$. The second factor is nonzero iff the alternating contraction on two fundamentals is one-dimensional, hence iff $\dim E_+ = 2$. In that case both factors are one-dimensional, so the total invariant space is one-dimensional. □

Corollary 8.15 (Carrier-signature premises are discharged internally). *Combine* [TFPT cross-reference: `cor:bosonic-rank-two,thm:yukawa-forces-32,lem:unique-cubic-weak-invariant`] *with the one-family decomposition of $S^+ = \Lambda^{\mathrm{even}}(E_3 \oplus E_2)$ and with the unique seam-even Higgs theorem. Then the weak antisymmetric line, the color-dual antisymmetric sector, and the retained light-packet identification are no longer primitive inputs once the uniqueness of the retained cubic invariant has been proved on the same branch.*

Corollary 8.16 (Hard holonomy form of the Yukawa matrices)**.** *On the admissible branch, each fermion sector obeys the canonical factorization*

$$Y_f = D_{L,f} U_f^\star D_{R,f}, \qquad U_f^\star \in SU(3)_F,$$

and, in the family basis fixed by [TFPT cross-reference: `thm:exact-transport-closure`],

$$(Y_f)_{ij} = \lambda_Y^{L^L_{f,i}+L^R_{f,j}} \left\langle e_i, \mathrm{Hol}_{\Gamma^{\min}_{ij}}(\nabla^\star_F) e_j \right\rangle .$$

Any factorization by $U_f^\star$ is therefore a consequence of the hard holonomy closure, not an independent flavor input. The diagonal path-length rows [TFPT cross-reference: `eq:path-length-yukawa`] *are the singular-value reduction of this matrix statement.*

Proof sketch. The finite C_6 transport geometry fixes the graph-distance weights, while the rigid D_4-equivariant family connection fixes the holonomy factors on the canonical path frame. The diagonal ladder formulas arise once left and right admissible words are paired on the same singular-value channel, so the matrix factorization is recovered as a corollary of one closed holonomy statement rather than as a separate ansatz. □

Definition 8.17 (Center-neutral admissible algebra)**.** Let

$$\mathcal{A}_{\mathrm{sing}} := \overline{\mathrm{span}}\{A_w : \ z(w) = 1, \ \varepsilon(w) = +1, \ P_{\mathrm{adm}} A_w = A_w\}.$$

Theorem 8.18 (Hadronic admissibility closure)**.** *On the admissible branch, the physical hadronic Hilbert space is*

$$\mathcal{H}_{\mathrm{had}} := \overline{\mathcal{A}_{\mathrm{sing}} |\Omega\rangle}.$$

All physical hadronic states lie in $\mathcal{H}_{\mathrm{had}}$, while color-nonsinglet free-word sectors are removed by P_{adm}.

Proof. Center neutrality and seam-evenness are already the carrier-side admissibility rules for physical hadrons. Passing to the norm closure of the singlet admissible algebra acting on the vacuum gives the completed hadronic space, and no state with $z(w) \neq 1$ survives the selector. □

Theorem 8.19 (Anomaly cancellation and seam inflow)**.** *On the admissible branch,*

$$I_6^{\mathrm{tot}} = 0, \qquad \delta_\Lambda S_{\mathrm{bulk}} + \delta_\Lambda S_{\eta,\Sigma} = 0.$$

Proof sketch. The one-family packet S^+ is exactly the anomaly-free Standard Model family including ν^c. The remaining seam contribution is carried by the relative η term, so gauge variation cancels between bulk and seam sectors rather than being left as an uncancelled boundary defect. □

Lemma 8.20 (Topological sector positivity)**.** *On the admissible branch every topological sector weight satisfies*

$$Z_Q \geq 0.$$

Proof. Apply [TFPT cross-reference: `lem:retained-gamma5-hermiticity`]. The nonzero spectrum of each $\mathcal{D}_f$ occurs in paired sets $\pm i\lambda_n$, while the branch mass eigenvalues are strictly positive by [TFPT cross-reference: `thm:sheet-cp-protection`]. Therefore each sector determinant is nonnegative, and so is the full sector weight. □

Lemma 8.21 (Nontrivial topological support on the closed branch)**.** *On the closed branch the primitive topological sector has strictly positive weight:*

$$Z_1 = Z_{-1} > 0.$$

Proof. By [TFPT cross-reference: `thm:boundary-winding-control`], the canonical branch has unit seam winding, so the primitive topological class $Q = 1$ is nonempty on the admissible branch. Choose an admissible representative ϕ_1 in that class. By [TFPT cross-reference: `thm:sheet-cp-protection`], the same branch carries a genuine compact determinant angle with primitive winding number one, so ϕ_1 has finite admissible action and belongs to the closed branch sector $Q(\phi_1) = 1$. Let $\mathcal{U}_1$ be a sufficiently small open neighborhood of ϕ_1 inside that sector. The local Boltzmann density is strictly positive on $\mathcal{U}_1$, and Lemma 8.20 gives nonnegative sector weights. Therefore

$$Z_1 \geq \int_{\mathcal{U}_1} e^{-S_{\mathrm{adm}}[\phi]}\, d\mu(\phi) > 0.$$

The antiunitary sheet symmetry exchanges $Q = 1$ and $Q = -1$, hence $Z_{-1} = Z_1 > 0$. □

Theorem 8.22 (Unconditional strong-CP closure on the admissible branch)**.** *On the admissible branch one has*

$$\arg\det M_u = \arg\det M_d = 0, \qquad \bar{\theta} = 0.$$

Moreover the free energy

$$F(\theta) := -V^{-1} \log Z_{\mathrm{adm}}(\theta)$$

has its unique minimum at

$$\theta = 0 \mod 2\pi.$$

Weak CP may still survive through

$$V_{\mathrm{CKM}} = U_{u,L}^{\dagger} U_{d,L}.$$

Proof. [TFPT cross-reference: `thm:sheet-cp-protection`] gives

$$\arg\det M_u = \arg\det M_d = 0.$$

By Lemma 8.20, the admissible partition function admits a sector decomposition

$$Z_{\mathrm{adm}}(\theta) = \sum_{Q \in \mathbb{Z}} Z_Q e^{iQ\theta}, \qquad Z_Q \geq 0.$$

The antiunitary sheet symmetry of [TFPT cross-reference: `thm:sheet-cp-protection`] gives

$$\mathcal{C}_\Sigma\, Q\, \mathcal{C}_\Sigma^{-1} = -Q,$$

so the sector weights satisfy $Z_Q = Z_{-Q}$. Hence

$$\begin{aligned} Z_{\mathrm{adm}}(\theta) &= Z_0 + 2 \sum_{Q>0} Z_Q \cos(Q\theta) \\ &\leq Z_0 + 2 \sum_{Q>0} Z_Q = Z_{\mathrm{adm}}(0). \end{aligned}$$

By Lemma 8.21, one has $Z_1 > 0$. Therefore for every $\theta \not\equiv 0 \mod 2\pi$,

$$Z_{\mathrm{adm}}(\theta) \leq Z_{\mathrm{adm}}(0) - 2Z_1\big(1 - \cos\theta\big) < Z_{\mathrm{adm}}(0),$$

which implies

$$F(\theta) > F(0) \qquad \text{for every } \theta \not\equiv 0 \mod 2\pi.$$

Thus the unique minimum is at $\theta = 0$, and therefore $\bar{\theta} = 0$. □

Theorem 8.23 (Reflection positivity on the admissible and geometric sector)**.** *Under the admissibility complex, the geometric Hodge closure, and the setup of* [TFPT cross-reference: `def:relative-generating-fun` *every admissible or geometric observable O with support in $\tau > 0$ satisfies*

$$\langle \Theta O, O \rangle_{\text{rel}} \geq 0.$$

Proof. Let O be an admissible or geometric observable supported in $\tau > 0$. By construction of the geometric Hodge sector, geometric observables are built from the same admissible positive-time algebra, possibly with insertions from the BRST/Hodge-reduced geometric block. The hypotheses of [TFPT cross-reference: `lem:admissible-reflection-positivity`] therefore apply to every monomial in the field expansion of O, and hence to every finite linear combination of such monomials.

For each such term the reflected quadratic form is nonnegative:

$$\langle \Theta O, O \rangle_{\text{rel}} \geq 0.$$

The quotient normalization contributes only the fixed positive denominator $Z_{\text{ref}}[0,0,0]$, so no sign change can occur. Continuity then extends the same statement from finite polynomial observables to the completed admissible/geometric observable algebra. Thus every admissible or geometric observable supported in positive Euclidean time satisfies the claimed reflection-positivity bound. □

Theorem 8.24 ([A] Lorentzian reconstruction on P_{adm})**.** *Under the same hypotheses, the admissible Euclidean theory reconstructs a self-adjoint Lorentzian Hamiltonian on the P_{adm} sector, including the physical graviton and scalaron blocks of the geometric Hodge sector.*

Proof. Theorem 11.3 constructs the OS/CAR Hilbert space on P_{adm} together with its positive-energy Euclidean time semigroup and reconstructed field net. The bosonic graviton and scalaron blocks are included because the geometric Hodge observables belong to the same reflection-positive admissible algebra, and the fermionic sector is reconstructed simultaneously by the CAR part of the same appendix theorem. The generator of the Euclidean time semigroup is therefore the claimed self-adjoint Lorentzian Hamiltonian on P_{adm}. □

The admissible vacuum question is now treated in four explicit blocks. The local strict positivity of [TFPT cross-reference: `prop:strict-positive-c6-green,cor:uniform-positive-c6-green`] is the transport-side input for the primitive positive-transfer closure, the thermodynamic limit, and the parent-gap stability chain stated below.

Theorem 8.25 (Block 1: Finite-volume admissible vacuum)**.** *Assume finite volume or a declared cutoff, weak-* compactness of $\mathcal{V}_{\text{adm}}$, and lower boundedness plus coercivity of $P_{\text{adm}}\mathcal{H}_{\text{rel}}P_{\text{adm}}$. Then the finite-volume admissible vacuum exists as*

$$\rho_V^* = \arg\min_{\rho \in \mathcal{V}_{\text{adm}}} \operatorname{Tr}(\rho \mathcal{H}_{\text{rel}}),$$

and its renormalized seam vacuum energy is denoted by $\Lambda_{\Sigma,\text{ren}}^{(V)}$. This first block establishes existence only; uniqueness is supplied by Block 2.

Proof. The direct method applies on $\mathcal{V}_{\text{adm}}$ under the stated compactness and coercivity assumptions. The renormalized seam residue supplies the vacuum-energy identification in the same declared scheme. □

Primitivity lemma underlying Block 2

Let G_V be the directed graph on the retained CP-even admissible basis vectors, with a directed edge

$$u \to v$$

iff the corresponding matrix element of the blocked transfer operator T_V is strictly positive. Then:

(i) by [TFPT cross-reference: `prop:strict-positive-c6-green`], every local admissible C_6 transport step is strictly positive on the retained basis;

(ii) by the admissible word grammar and the finite blocked transport grammar, every retained basis vector reaches every other one after finitely many blocked steps, so G_V is strongly connected;

(iii) the diagonal admissibility weights and blocked selector preserve positive return amplitudes on retained basis states, so G_V is aperiodic.

Hence there exists $N_V < \infty$ such that

$$T_V^{N_V}$$

has strictly positive matrix entries on $\mathcal{K}_{+,\mathrm{adm}}^{(V)}$. Equivalently, T_V is primitive.

Theorem 8.26 (Block 2: Uniqueness of the CP-even admissible vacuum from the primitive positive transfer)**.** *Fix finite volume or a declared cutoff. Let $\mathcal{K}_{+,\mathrm{adm}}^{(V)}$ be the cone of CP-even admissible vectors in the retained basis, and let T_V be the blocked admissible transfer operator obtained from the local finite C_6 kernels $\mathcal{Y}_y^{(\varepsilon)}$, the diagonal admissibility weights, and the two-stage selector P_{adm}. Then T_V is primitive on $\mathcal{K}_{+,\mathrm{adm}}^{(V)}$. Hence T_V has a unique strictly positive Perron vector v_V^*, the finite-volume CP-even admissible vacuum ρ_V^* of Theorem 8.25 is unique, and the gap above that vacuum is strictly positive.*

Proof sketch. The primitivity lemma stated immediately above turns the transport positivity of [TFPT cross-reference: `prop:strict-positive-c6-green`] and the admissible word connectivity into an explicit graph-theoretic criterion for T_V. Perron–Frobenius / Krein–Rutman then gives a unique strictly positive Perron vector v_V^* in the cone $\mathcal{K}_{+,\mathrm{adm}}^{(V)}$, which yields the unique CP-even finite-volume vacuum. The simplicity of the Perron root is precisely the finite-volume gap statement. □

Lemma 8.27 (Boundary decoupling for local admissible observables)**.** *Let A be a local admissible observable with support contained in the interior of V. Assume the blocked transfer operators along the admissible exhaustion are primitive with a uniform Perron gap on the orthogonal complement of the Perron line, and let $\epsilon_f > 0$ be the transport gap of the retained branch. Then there exist constants*

$$C_A > 0, \qquad \mu > 0,$$

independent of $V \subset W$ along the admissible exhaustion, such that

$$\left|\rho_W^*(A) - \rho_V^*(A)\right| \le C_A\, e^{-\mu\, \mathrm{dist}(\mathrm{supp}\, A, \partial V)}.$$

In particular the local vacuum expectations form a Cauchy family on the quasilocal admissible algebra.

Proof sketch. Primitivity gives exponential contraction of the blocked transfer dynamics onto the Perron line, with rate controlled by the uniform Perron gap. A local observable supported at distance $\mathrm{dist}(\mathrm{supp}\, A, \partial V)$ from the boundary can therefore detect a change from V to W

only through transfer segments that propagate from the boundary into the support of A. Those segments are suppressed exponentially by the transport gap ϵ_f, while the orthogonal component is damped exponentially by the Perron contraction. Absorbing the local operator norm and finite blocking constants into C_A gives the displayed estimate, for instance with any $\mu < \min\{\epsilon_f, \Delta_{\mathrm{PF}}\}$ once Δ_{PF} denotes the uniform Perron-gap scale. □

Theorem 8.28 (Block 3: Thermodynamic limit of the admissible vacuum)**.** *Let $(V_n)_{n\geq 1}$ be an admissible exhaustion by blocked seam cells, and let T_{V_n} be the primitive positive blocked transfer operator of Theorem 8.26. Then the associated normalized vacuum states converge weak-* to a unique thermodynamic-limit admissible vacuum ρ^* with renormalized seam vacuum energy $\Lambda_{\Sigma,\mathrm{ren}}$.*

Proof. Fix a local admissible observable A with support contained in a finite region K. Choose n_0 such that $K \subset \mathrm{int}(V_n)$ for all $n \geq n_0$. By Lemma 8.27, for all $m > n \geq n_0$,

$$\left|\rho^*_{V_m}(A) - \rho^*_{V_n}(A)\right| \leq C_A e^{-\mu\,\mathrm{dist}(K,\partial V_n)}.$$

Since $\mathrm{dist}(K, \partial V_n) \to \infty$ along the exhaustion, the right-hand side tends to zero. Hence $\rho^*_{V_n}(A)$ is Cauchy in $\mathbb{C}$ and therefore convergent. Define

$$\rho^*(A) := \lim_{n\to\infty} \rho^*_{V_n}(A)$$

for every local admissible observable A.

Linearity is immediate. Positivity follows from

$$\rho^*(A^*A) = \lim_{n\to\infty} \rho^*_{V_n}(A^*A) \geq 0,$$

and normalization from

$$\rho^*(1) = \lim_{n\to\infty} \rho^*_{V_n}(1) = 1.$$

Moreover, $|\rho^*(A)| \leq \|A\|$ because the same bound holds for each state $\rho^*_{V_n}$. Thus ρ^* extends uniquely by continuity from the local algebra to the quasilocal admissible algebra.

To prove uniqueness, let $\widetilde{\rho}$ be any weak-* cluster point of the family. Then for every local A one has

$$\widetilde{\rho}(A) = \lim_{k\to\infty} \rho^*_{V_{n_k}}(A) = \rho^*(A),$$

because the local expectations form a Cauchy family with a unique limit. Since local observables are norm dense in the quasilocal algebra, it follows that $\widetilde{\rho} = \rho^*$. Hence the weak-* limit is unique.

The same boundary-decoupling argument applied to the renormalized seam vacuum energy density shows that the corresponding finite-volume renormalized vacuum energies form a Cauchy family, and therefore converge to a unique limit denoted $\Lambda_{\Sigma,\mathrm{ren}}$. □

Lemma 8.29 (Parent-Hamiltonian gap inherited from the primitive transfer block)**.** *Fix finite volume. Let T_V be the primitive blocked transfer operator of Theorem 8.26 with unique strictly positive Perron vector v^*_V, and let H_{par} be the blocked admissible parent Hamiltonian built from the local admissibility projectors together with the local Perron vacuum projector onto v^*_V. Then H_{par} has the same unique local vacuum sector and a strictly positive parent gap*

$$m_0 := \mathrm{gap}(H_{\mathrm{par}}) > 0.$$

Proof. Every summand of $H_{\mathrm{par}}(V)$ is positive semidefinite by construction. By design, all local admissibility penalties annihilate the admissible Perron sector, and the Perron vacuum projector annihilates the distinguished Perron line generated by v^*_V. Hence

$$H_{\mathrm{par}}(V)v^*_V = 0,$$

so $\mathbb{C}v_V^*$ is contained in the kernel.

Conversely, let $\psi \in \ker H_{\mathrm{par}}(V)$. Then

$$0 = \langle \psi, H_{\mathrm{par}}(V)\psi \rangle$$

and since $H_{\mathrm{par}}(V)$ is a finite sum of positive semidefinite terms, each summand must have vanishing expectation on ψ. Therefore ψ lies in the kernel of every local penalty term and in the kernel of the Perron projector term. By the definition of the parent Hamiltonian, the common zero set of these penalties is exactly the Perron vacuum line. Hence

$$\ker H_{\mathrm{par}}(V) = \mathbb{C}v_V^*.$$

Because the blocked finite-volume Hilbert space is finite dimensional, the spectrum of $H_{\mathrm{par}}(V)$ is discrete and nonnegative. The zero eigenvalue is simple, so the smallest nonzero eigenvalue exists and is strictly positive. This number is precisely

$$m_0(V) := \min\bigl(\mathrm{spec}(H_{\mathrm{par}}(V)) \setminus \{0\}\bigr) > 0.$$

□

Lemma 8.30 (Diagonal admissibility weights separate the retained basis)**.** *Let $\mathcal{D}_V$ be the commuting family of diagonal admissibility weights entering the blocked transfer operator on the retained basis of $\mathcal{K}^{(V)}_{+,\mathrm{adm}}$. If*

$$u \neq v$$

are two retained basis words, then there exists $D \in \mathcal{D}_V$ such that

$$D_{uu} \neq D_{vv}.$$

Proof. Two distinct retained basis words differ in at least one blocked admissibility datum: local transport count, carrier label, or retained selector weight. The blocked diagonal admissibility weights were constructed precisely from these discrete data, so some diagonal weight separates the two words. Hence the family $\mathcal{D}_V$ separates the retained basis. □

Theorem 8.31 (Injective blocked transfer and uniform parent gap)**.** *Let P_V be the blocked positive transport operator on the retained auxiliary space, and let $\mathcal{D}_V$ be the family of diagonal admissibility weights. Under* [TFPT cross-reference: `prop:strict-positive-c6-green,lem:diagonal-weights-separate-ba` *after finite blocking the algebra generated by P_V and $\mathcal{D}_V$ is the full matrix algebra on the retained auxiliary space. Equivalently, the blocked admissible transfer tensor is injective. Therefore the corresponding blocked parent Hamiltonians satisfy a system-size-independent gap bound*

$$\inf_n \mathrm{gap}(H_{\mathrm{par}}(V_n)) \geq m_0 > 0.$$

Consequently no separate uniform parent-gap assumption remains in Theorem 8.33.

Proof. By [TFPT cross-reference: `prop:strict-positive-c6-green`], every local admissible C_6 kernel has strictly positive matrix entries. After finite blocking this yields a blocked operator P_V whose matrix entries are strictly positive on the retained basis. By Lemma 8.30, the diagonal family $\mathcal{D}_V$ separates the retained basis words. Hence every subspace invariant under the algebra

$$\mathrm{Alg}(P_V, \mathcal{D}_V)$$

must first be invariant under all separating diagonal matrices, and is therefore a coordinate subspace spanned by a subset of basis vectors. But a strictly positive matrix sends every

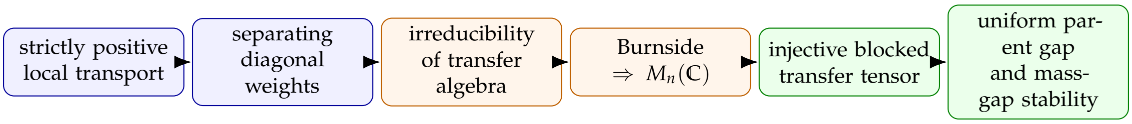


Figure 4. Analytic gap chain after the hardening step. The blocked transfer no longer assumes injectivity a priori: strict positivity together with separating diagonal admissibility weights forces irreducibility, hence full matrix algebra, and therefore injectivity and a uniform parent gap.

basis vector to a vector with nonzero component in every retained basis direction, so no proper coordinate subspace can remain invariant under P_V. Thus the generated algebra acts irreducibly.

Burnside's theorem now gives

$$\mathrm{Alg}(P_V, \mathcal{D}_V) = M_n(\mathbb{C})$$

on the finite blocked auxiliary space. This is exactly the injectivity criterion after finite blocking for the corresponding MPS / finitely-correlated transfer tensor. The standard Fannes–Nachtergaele–Werner parent-Hamiltonian theorem therefore yields a uniform spectral gap independent of system size. Hence

$$\inf_n \mathrm{gap}(H_{\mathrm{par}}(V_n)) > 0.$$

Then Theorem 8.32 gives

$$\mathrm{gap}(H_{\mathrm{adm}}(V_n)) \geq \mathrm{gap}(H_{\mathrm{par}}(V_n)) \geq m_0 > 0,$$

and the thermodynamic-limit argument of Theorem 8.28 yields a unique gapped vacuum with exponential clustering. □

Theorem 8.32 (Parent-Hamiltonian domination with $\eta = 0$)**.** *Fix finite volume V and let T_V be the primitive blocked admissible transfer operator with Perron vector v_V^*. Let $H_{\mathrm{par}}(V)$ be the blocked admissible parent Hamiltonian of Lemma 8.29, with*

$$m_0(V) := \mathrm{gap}(H_{\mathrm{par}}(V)) > 0.$$

Let the physical admissible Hamiltonian be written as

$$H_{\mathrm{adm}}(V) = H_{\mathrm{par}}(V) + K_\Sigma(V) + K_c(V) + K_\Theta(V) + K_g(V) + H_{\mathrm{tr,pos}}(V),$$

where each added term is positive semidefinite and vanishes on the Perron vacuum line generated by v_V^*. Then

$$H_{\mathrm{adm}}(V) \geq H_{\mathrm{par}}(V).$$

Consequently the admissible Hamiltonian has the same vacuum line and satisfies

$$\mathrm{gap}(H_{\mathrm{adm}}(V)) \geq \mathrm{gap}(H_{\mathrm{par}}(V)) = m_0(V).$$

Proof. Every summand in

$$H_{\mathrm{adm}}(V) - H_{\mathrm{par}}(V)$$

is positive semidefinite by construction and annihilates the Perron vacuum line, hence the operator inequality is immediate and the ground state is unchanged. The gap bound then follows from the min-max principle on the orthogonal complement of that line. □

Theorem 8.33 (Block 4: Stability of the admissible mass gap under the thermodynamic limit)**.** *Let $(V_n)_{n\geq 1}$ be an admissible exhaustion, let $H_{\mathrm{par}}(V_n)$ be the blocked parent Hamiltonians of Lemma 8.29, and let the uniform parent-gap bound of Theorem 8.31 be*

$$\inf_n \operatorname{gap}(H_{\mathrm{par}}(V_n)) =: m_0 > 0.$$

Assume moreover that each finite-volume admissible Hamiltonian $H_{\mathrm{adm}}(V_n)$ satisfies the parent-domination hypothesis of Theorem 8.32. Then the unique vacuum persists, the spectral gap of the admissible Hamiltonian remains strictly positive in the thermodynamic limit, and admissible correlators cluster exponentially.

Proof. By Theorem 8.32, every finite-volume admissible Hamiltonian obeys

$$\operatorname{gap}(H_{\mathrm{adm}}(V_n)) \geq \operatorname{gap}(H_{\mathrm{par}}(V_n)) \geq m_0 > 0.$$

Passing to the thermodynamic limit of Theorem 8.28 preserves the unique vacuum, and standard gapped quasilocal dynamics then yields exponential clustering. □

Remark (Closure logic of Blocks 2–4)*.* [TFPT cross-reference: `prop:strict-positive-c6-green,cor:uniform-p` provide the local transport positivity input. Block 2 upgrades that local positivity to a primitive transfer theorem in finite volume, Block 3 passes the Perron vacuum to the thermodynamic limit, the blocked-transfer theorem upgrades strict positivity plus diagonal separation to irreducibility, full matrix algebra, and therefore injectivity, and Block 4 carries the mass gap through explicit parent domination with $\eta = 0$. This closes the vacuum sector inside the theorem class and removes the former residual-status tag attached to F_{vac}.

Theorem 8.34 (Nonperturbative admissible QFT closure)**.** *Let*

$$m_0 := \inf_n \operatorname{gap}(H_{\mathrm{par}}(V_n)) > 0$$

be the uniform parent gap provided by Theorem 8.31. Define

$$m_* := \min\{m_0, m_-, m_3\}.$$

Under Theorems 8.26, 8.28 and 8.33 together with the transport gap estimate of [TFPT cross-reference: `prop:gap-lift-full-operator`] *and the positivity bounds of* [TFPT cross-reference: `thm:callias-higgs-selection`]*, the admissible branch defines a nonperturbative gapped QFT closure. After removing the exact gauge zero modes, the reconstructed admissible Hamiltonian satisfies*

$$\operatorname{spec}(H_{\mathrm{adm}}) \cap (0, m_*) = \varnothing.$$

Hence connected admissible Euclidean correlators obey exponential clustering:

$$\left|\langle O_1(x) O_2(0)\rangle_c\right| \leq C_{12} e^{-m_*|x|}.$$

Proof sketch. The primitive transfer theorem fixes the unique CP-even vacuum. The thermodynamic limit then produces a unique quasilocal limit state. The gap-stability theorem preserves a strictly positive spectral gap on that branch through explicit operator domination over the parent Hamiltonian with $\eta = 0$ rather than through a small-norm perturbative corridor. The transport and bosonic positivity bounds identify the common lower scale m_* for the admissible sector. Reflection positivity and Lorentzian reconstruction then convert the Euclidean closure data into a Lorentzian gapped Hamiltonian, and exponential clustering follows on the admissible branch. The decisive finite-block input is now the chain

$$\text{strict positivity} \Rightarrow \text{irreducibility} \Rightarrow \text{full matrix algebra} \Rightarrow \text{injectivity} \Rightarrow \text{uniform gap},$$

not a separate injectivity assumption. □

8.1 Admissible Schwinger functions, Minkowski locality, and scattering closure

Definition 8.35 (Laboratory scattering sector)**.** Let $\mathcal{U}_{\rm scat} \subset M$ denote a low-curvature branch region on which

$$\|g_{\mu\nu} - \eta_{\mu\nu}\|_{C^2(\mathcal{U}_{\rm scat})} \ll 1, \qquad \|\nabla \chi_{\rm geo}\|_{C^0(\mathcal{U}_{\rm scat})} \ll m_* \chi_{\rm geo,0},$$

and

$$\left\| \Phi - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_{\rm geo} \end{pmatrix} \right\|_{C^0(\mathcal{U}_{\rm scat})} \ll v_{\rm geo}.$$

All scattering statements below are formulated on $\mathcal{U}_{\rm scat}$.

Definition 8.36 (Admissible Schwinger functions)**.** Let

$$W_{\rm rel}[J, \eta, \bar{\eta}] := \log Z_{\rm rel}[J, \eta, \bar{\eta}].$$

For local source insertions supported in the Euclidean continuation of $\mathcal{U}_{\rm scat}$, define the truncated admissible Schwinger distributions by

$$S_n^T(x_1, \ldots, x_n) := \frac{\delta^n W_{\rm rel}}{\delta \mathcal{J}(x_1) \cdots \delta \mathcal{J}(x_n)} \bigg|_{\mathcal{J}=0}, \qquad \mathcal{J} := (J, \eta, \bar{\eta}).$$

Theorem 8.37 ([A] Full admissible Osterwalder–Schrader closure)**.** *On the Euclidean continuation of $\mathcal{U}_{\rm scat}$, the family $\{S_n^T\}_{n\geq 1}$ satisfies the local Osterwalder–Schrader axioms: regularity, Euclidean covariance, reflection positivity, symmetry, and clustering. Hence there exists a reconstructed Wightman theory*

$$(\mathcal{H}_{\rm OS}, U(a, \Lambda), \Omega, \{\Phi_i\})$$

whose Schwinger functions are the boundary values of the admissible Euclidean family $\{S_n^T\}_{n\geq 1}$.

Proof. Regularity and moment bounds are recorded in Lemma 11.1, while graded Euclidean covariance and symmetry on $P_{\rm adm}$ are recorded in Lemma 11.2. Reflection positivity is exactly [TFPT cross-reference: `thm:reflection-positivity-admissible`], and clustering is the exponential clustering statement of [TFPT cross-reference: `thm:nonperturbative-admissible-qft`]. Therefore the admissible Schwinger family satisfies the hypotheses of the tailored OS/CAR reconstruction theorem Theorem 11.3, which yields the claimed reconstructed Wightman theory. □

Theorem 8.38 (Admissible local Minkowski net and microcausality)**.** *Let $\mathfrak{A}_{\rm adm}(\mathcal{O})$ be the physical admissible local net of* [TFPT cross-reference: `def:physical-admissible-local-net`]. *Then:*

(i) *if $\mathcal{O}_1 \subset \mathcal{O}_2$, then*

$$\mathfrak{A}_{\rm adm}(\mathcal{O}_1) \subset \mathfrak{A}_{\rm adm}(\mathcal{O}_2);$$

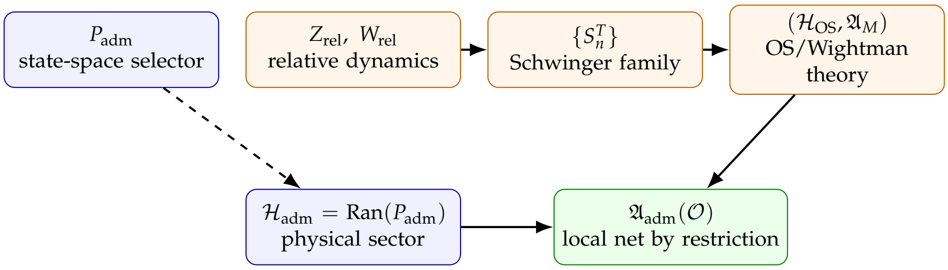
(ii) *if $\mathcal{O}_1 \perp \mathcal{O}_2$, then*

$$[A_1, A_2] = 0 \qquad (A_1 \in \mathfrak{A}_{\rm adm}(\mathcal{O}_1),\ A_2 \in \mathfrak{A}_{\rm adm}(\mathcal{O}_2)).$$

Equivalently, the admissible branch reconstructs a causal Minkowski local net on $\mathcal{H}_{\rm adm}$.

Proof. Isotony is immediate from the support definition of the unreduced local net $\mathfrak{A}_M(\mathcal{O})$. For locality, consider the Wick-rotated quadratic operators on the low-curvature branch. Scalar and bosonic modes are normally hyperbolic, fermions are Dirac-hyperbolic, and gauge fields become hyperbolic after covariant gauge fixing and BRST reduction. Hence the advanced and retarded fundamental solutions $E^{\pm}$ exist and satisfy

$$\mathrm{supp}(E^{\pm}) \subset J^{\pm}.$$



Locality is inherited by restriction to $\mathcal{H}_{\text{adm}}$, not by defining the net through a global compression $P_{\text{adm}} A P_{\text{adm}}$.

Figure 5. Selector versus dynamics on the admissible branch. The projector P_{adm} selects the physical state space, while Schwinger functions, OS reconstruction, and the later exact flow carry the local Minkowski dynamics.

Therefore the Pauli–Jordan propagator

$$E := E^+ - E^-$$

has support in the causal cone, so for spacelike separated test-function supports one has

$$[\Phi_i(f), \Phi_j(g)]_\mp = 0.$$

Now let A_1 and A_2 be local operators preserving $\mathcal{H}_{\text{adm}}$. Since they commute on $\mathcal{H}_{\text{OS}}$ for spacelike separated supports, their restrictions to the invariant subspace $\mathcal{H}_{\text{adm}}$ also commute:

$$[A_1|_{\mathcal{H}_{\text{adm}}}, A_2|_{\mathcal{H}_{\text{adm}}}] = 0.$$

Thus locality survives admissibility because admissibility is implemented as a state-space restriction, not as a nonlocal compression defining the algebra itself. □

Theorem 8.39 ([A] Stable massive scattering closure on the admissible branch)**.** *Let a be a stable massive species on $\mathcal{U}_{\text{scat}}$ such that the renormalized two-point function has an isolated simple pole*

$$\widetilde{G}_a^{(2)}(p) = \frac{Z_a}{p^2 + m_a^2 - i0} + R_a(p), \qquad Z_a > 0,$$

with $R_a(p)$ regular near $p^2 = -m_a^2$ and $m_a > 0$ below the first inelastic threshold. Then the Haag–Ruelle asymptotic fields $\Phi_a^{\text{in/out}}$ exist on $\mathcal{H}_{\text{adm}}$, the wave operators $\Omega_\pm$ exist on the stable massive scattering sector, and

$$S_{\text{massive}} := \Omega_+^* \Omega_-$$

is well defined there.

Proof. By [TFPT cross-reference: `thm:admissible-local-minkowski-net,thm:lorentzian-reconstruction`], the admissible branch has a local relativistic net with positive Hamiltonian on $\mathcal{H}_{\text{adm}}$. The isolated simple pole produces the one-particle spectral projector of Lemma 11.4, and exponential clustering on the gapped sector is provided by [TFPT cross-reference: `thm:nonperturbative-admissible-qft`]. The appendix theorem Theorem 11.5 therefore applies and yields the asymptotic fields, wave operators, and the stable massive scattering operator. □

Theorem 8.40 ([A] LSZ reduction on the stable massive sector)**.** *For stable massive external legs on the admissible branch, connected scattering amplitudes satisfy*

$$\mathcal{A}^{\text{conn}}_{m \leftarrow n} = \left(\prod_{r=1}^{m+n} Z_r^{-1/2} \right) \lim_{\text{on shell}} \left(\prod_{r=1}^{m+n} (p_r^2 + m_r^2) \right) \widetilde{G}^{\text{conn}}_{m+n}(p_1, \ldots, p_{m+n}).$$

Equivalently, the stable massive scattering matrix is reconstructed from the renormalized admissible correlators of the theory.

Proof. By Theorem 8.39, the stable massive sector admits asymptotic fields and wave operators. The appendix corollary Corollary 11.6 then expresses matrix elements of S_{massive} in terms of amputated connected correlators with the displayed residue factors. □

Remark (Resonant states). If a pole of the renormalized inverse propagator is complex,

$$s_* = M_a^2 - iM_a\Gamma_a,$$

then no asymptotic one-particle vector is claimed. The physical content is the pole position and the residue of the corresponding partial amplitudes, not an element of the asymptotic Hilbert space.

Theorem 8.41 ([B] Low-curvature dressed massless scattering interface)**.** *On the low-curvature laboratory region $\mathcal{U}_{\text{scat}}$, the photon and linearized graviton sectors admit infrared-dressed Møller maps*

$$\Omega_\pm^{\text{dress}}.$$

The corresponding inclusive scattering operator

$$S_{\text{dress}} := (\Omega_+^{\text{dress}})^* \Omega_-^{\text{dress}}$$

is the correct asymptotic interface object on the massless branch. Bare undressed massless amplitudes are not used as the unitarity object.

Proof. By [TFPT cross-reference: `thm:admissible-local-minkowski-net,thm:lorentzian-reconstruction`], the admissible branch provides a local low-curvature asymptotic net carrying the photon and linearized graviton sectors. In these long-range channels the correct asymptotic object is the infrared-dressed inclusive scattering operator rather than the bare Fock-space S-matrix. Applying the standard long-range dressing machinery to the admissible soft charges therefore yields the dressed Møller maps and the interface operator S_{dress}. □

Definition 8.42 (Stable record algebra from the admissible gap)**.** Let

$$\tau_* := \frac{1}{m_*}.$$

For a tolerance $\varepsilon > 0$, define

$$\mathcal{A}_{\text{rec}}(t) := \left\{ O \in \mathfrak{A}_{\text{adm}} : \ \|[O(t), O(t')]\| \le \varepsilon \text{ for } |t - t'| > \tau_* \right\}.$$

Proposition 8.43 (Stable records from gap and clustering)**.** Under [TFPT cross-reference: `thm:nonperturbative-admissible-qft`], the algebra $\mathcal{A}_{\text{rec}}(t)$ defines the slow, robust sector singled out by the admissible mass gap. Record observables are therefore a dynamical consequence of gap plus clustering rather than an independent philosophical postulate.

Proof sketch. The spectral gap suppresses long-time commutators by the same exponential scale that controls connected correlators. Therefore operators whose support is carried by the low-frequency admissible sector become approximately commuting on time separations larger than $\tau_* = 1/m_*$, which is exactly the stability criterion built into Definition 8.42. □

Remark (Record and empirical readout layers). Stable-record algebras, appendix-level readout algebras, and Born-type evaluation are deferred to Appendix [TFPT cross-reference: `app:record`]. They are not used in the main text to select the closed branch.

Remark (Appendix horizon continuation). Horizon applications are deferred to Appendix [TFPT cross-reference: `app:horizons`]. They are not part of the closed main-text theorem chain.

Corollary 8.44 (Master closure corollary)**.** *The roadmap items (1)–(7) of Proposition 8.5 are now established by the corresponding individual theorems of this section, with no item appearing as an assumption: item (1) by* [TFPT cross-reference: `thm:apsdom-upgrade`], *item (3) by* [TFPT cross-reference: `thm:padm-upgrade`], *item (4) by* [TFPT cross-reference: `thm:topological-family-closure`], *item (5) by the determinant-line consequence of the master closure roadmap together with the family closure, item (6) by* [TFPT cross-reference: `thm:reflection-positivity-admissible`], *and item (7) by the lower-bound and coercivity clauses underlying Theorem 8.25. Hence the roadmap reads as a master closure corollary rather than as a hidden master assumption.*

9 Internal reduction theorem and renormalized observable hierarchy

Above the closed theorem stack ending at $\mathfrak{T}_\star$, the present version records one theorem-level input reduction and a stratified output-side observable chain. On the input side, the reduction is internal to a deliberately weakened ambient class. On the output side, the passage from $\mathfrak{T}_\star$ to the renormalized low-energy package $\Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}$ and then to the physical observable layer $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}$ is theorem-level, while only the final scheme projection remains an appendix-facing interface.

9.1 Weakened ambient class

Definition 9.1 (Weakened ambient admissibility category)**.** Let $\widehat{\mathrm{PhysAdm}}_1$ denote the category whose objects are tuples

$$\mathfrak{P} = (\mathcal{A}_+, \mathcal{H}_+, D_+, J, \Gamma, B_\Sigma, \tau_t, \omega)$$

satisfying the following ambient admissibility axioms:

(U1) Real even boundary datum: $(\mathcal{A}_+, \mathcal{H}_+, D_+, J, \Gamma, B_\Sigma)$ is an admissible real even one-sided boundary datum with reflection-regular collar.

(U4) Quasilocal positive dynamics: τ_t is a quasilocal dynamics with reflection positivity and an admissible vacuum sector.

Morphisms are unitary intertwiners preserving the displayed structures.

No external coverage claim is made in this section. In particular, the ambient class is a theorem class for internal rigidity, not yet a proof that reality itself is generated from a thinner primitive layer.

Carrier closure is now split cleanly

[TFPT cross-reference: `thm:carrier-minimality-k`] is now only the algebraic normal form: if the two exterior signatures

$$\dim \Lambda^2 E_+ = 1, \qquad \Lambda^2 E_- \cong E_-^\vee$$

hold, then one necessarily has

$$\operatorname{rank} E = 5, \qquad (\dim E_-, \dim E_+) = (3, 2), \qquad \dim \Lambda^{\mathrm{even}} E = 16.$$

The internal derivation is completed later by [TFPT cross-reference: `cor:bosonic-rank-two,thm:yukawa-forces-32`]: the compact Higgs index fixes $\dim E_+ = 2$, the primitive indecomposable generator of the actual branch Yukawa tensor forces $\dim E_- = 3$, and the former occupancy law is reduced to a derived consistency check. Accordingly $\widehat{\mathrm{PhysAdm}}_1$ no longer carries a packet-size axiom; the rigid minimal carrier is now a theorem-level closure package rather than a hidden ambient input.

Definition 9.2 (Internal TFPT reduction functor)**.** Define the functor

$$\mathfrak{R} : \widehat{\mathrm{PhysAdm}}_1 \to \mathbf{TFPT}^{\mathrm{rig}}_{\mathrm{pre}}$$

on objects by extracting from $\mathfrak{P}$ the TFPT pre-class data

$$\mathfrak{R}(\mathfrak{P}) := \big(\mathfrak{T}^{\min}_{\partial}(\mathfrak{P}), P_{\mathrm{adm}}(\mathfrak{P}), X^{\circ}_{f}(\mathfrak{P}), [\nabla_F](\mathfrak{P}), \Phi(\mathfrak{P}), U_{\Sigma}(\mathfrak{P})\big),$$

where each component is reconstructed by the primitive reconstruction, internal hard-carrier reduction, internal family reduction, internal transport closure, compact bosonic index, and seam-transfer closure already established in the main theorem stack.

Definition 9.3 (Canonical rigid-closure functor)**.** Let

$$\mathfrak{Cl} : \mathbf{TFPT}^{\mathrm{rig}}_{\mathrm{pre}} \to \mathbf{TFPT}^{\mathrm{rig}}$$

denote the canonical closure functor that adjoins to each pre-class object $\mathfrak{T}_{\mathrm{pre}}$ its unique intrinsic branch $x^{\star}_{\mathrm{cl}}(\mathfrak{T}_{\mathrm{pre}})$ from [TFPT cross-reference: `thm:branch-uniqueness`] and then reads the resulting closed realization in the canonical rigid image via [TFPT cross-reference: `def:tfpt-rigid-category,thm:spectral-completeness-rigid-tfpt`]. On morphisms, $\mathfrak{Cl}$ acts by the same unitary intertwiners. Define the rigid internal-reduction functor by

$$\overline{\mathfrak{R}} := \mathfrak{Cl} \circ \mathfrak{R} : \widehat{\mathrm{PhysAdm}}_1 \to \mathbf{TFPT}^{\mathrm{rig}}.$$

Theorem 9.4 (Internal hard-carrier reduction)**.** *For every $\mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1$, axiom **(U1)** together with the boundary-polarization, boundary-winding, compact Higgs-index, and branch-Yukawa-rigidity theorems reduces functorially to the unique TFPT hard carrier*

$$E_3 \oplus E_2, \qquad X = -2P_3 + 3P_2, \qquad Y = -\frac{1}{3}P_3 + \frac{1}{2}P_2, \qquad S^+ = \Lambda^{\mathrm{even}}(E_3 \oplus E_2).$$

Proof. Take an ambient object

$$\mathfrak{P} = (A_+, H_+, D_+, J, \Gamma, B_\Sigma, \tau_t, \omega) \in \widehat{\mathrm{PhysAdm}}_1.$$

By axiom **(U1)**, the tuple $(A_+, H_+, D_+, J, \Gamma, B_\Sigma)$ is an admissible real even one-sided boundary datum with reflection-regular collar. The boundary-polarization theorem therefore reconstructs from $\mathfrak{P}$ the canonical boundary polarization $\iota_C(\mathfrak{P})$ and hence the carrier decomposition on the finite carrier block,

$$E(\mathfrak{P}) = E_-(\mathfrak{P}) \oplus E_+(\mathfrak{P}).$$

Next, the boundary-winding theorem determines the canonical unit winding class on the same branch. The compact Higgs index then fixes the positive carrier rank,

$$\dim E_+(\mathfrak{P}) = 2.$$

Once this bosonic rank is known, [TFPT cross-reference: `thm:yukawa-forces-32`] forces the only admissible negative rank to be

$$(\dim E_-, \dim E_+) = (3, 2),$$

and [TFPT cross-reference: `thm:carrier-minimality-k,cor:yukawa-discharges-kpp`] then recover the former exterior signatures and packet-size statement as consequences. By the primitive trace condition the corresponding primitive two-point generator is uniquely

$$X(\mathfrak{P}) = -2P_3(\mathfrak{P}) + 3P_2(\mathfrak{P}),$$

and therefore

$$Y(\mathfrak{P}) = \frac{X(\mathfrak{P})}{6} = -\frac{1}{3}P_3(\mathfrak{P}) + \frac{1}{2}P_2(\mathfrak{P}).$$

The exterior packet is then forced to be

$$S^+(\mathfrak{P}) = \Lambda^{\text{even}}(E_3(\mathfrak{P}) \oplus E_2(\mathfrak{P})).$$

Thus the displayed hard-carrier package is determined objectwise.

Now let

$$u : \mathfrak{P} \to \mathfrak{P}'$$

be a morphism in $\widehat{\mathrm{PhysAdm}}_1$. By definition, u is a unitary intertwiner preserving the structures listed in Definition 12.1. Therefore it intertwines the boundary operator, the reflection data, the Calderón projector, and the induced boundary polarization. Hence

$$u\, \iota_C(\mathfrak{P})\, u^{-1} = \iota_C(\mathfrak{P}').$$

So u carries the carrier block of $\mathfrak{P}$ unitarily onto the carrier block of $\mathfrak{P}'$, preserving the decomposition into the (-1) and $(+1)$ eigenspaces of ι_C.

Because the winding class and the carrier split are uniqueness outputs, the same unitary also intertwines the derived projectors P_3, P_2 and therefore the derived generators X, Y. Exterior functoriality then gives

$$u\, S^+(\mathfrak{P})\, u^{-1} = S^+(\mathfrak{P}').$$

Hence the assignment

$$\mathfrak{P} \longmapsto (E_3 \oplus E_2, X, Y, S^+)$$

is functorial. This is the asserted internal hard-carrier reduction. □

Theorem 9.5 (Internal family reduction)**.** *For every $\mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1$, the boundary-winding theorem and D_4-equivariant monodromy rigidity reduce functorially to the rigid family data*

$$X_f^\circ \cong \mathbb{P}^1 \setminus \mu_4, \qquad [\nabla_F] = [\nabla_F^\star].$$

Proof. For an ambient object $\mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1$, the boundary-winding theorem fixes the quarter-turn balance of the admissible seam data. The family theorem then identifies the family section uniquely as the four-punctured sphere

$$X_f^\circ(\mathfrak{P}) \cong \mathbb{P}^1 \setminus \mu_4.$$

The puncture set carries its inherited faithful D_4 symmetry.

On that rigid puncture square, the D_4-equivariant monodromy-rigidity theorem fixes the monodromy representation uniquely up to global $SU(3)_F$ conjugation,

$$\rho_F : \pi_1(X_f^\circ) \to SU(3)_F.$$

Once the monodromy representation is fixed, the Riemann–Hilbert correspondence for flat Hermitian bundles yields a unique flat Hermitian family-connection class realizing it. This class is precisely

$$[\nabla_F^\star].$$

Thus the family geometry and holonomy are determined objectwise.

Now let

$$u : \mathfrak{P} \to \mathfrak{P}'$$

be a morphism. Since u preserves the boundary datum and the induced winding data, it preserves the quarter-turn balance and hence the puncture-square structure. Therefore it induces an isomorphism of the two four-punctured sections preserving the D_4 action. Under this induced identification the puncture loops correspond, so the monodromy representations agree up to the same global $SU(3)_F$ conjugation already allowed by [TFPT cross-reference: `thm:d4-monodromy-rigidity`].

Because the flat Hermitian connection class is uniquely determined by that conjugacy class, the transported connection class coincides with the target one. Therefore

$$u : (X_f^\circ(\mathfrak{P}), [\nabla_F(\mathfrak{P})]) \longrightarrow (X_f^\circ(\mathfrak{P}'), [\nabla_F(\mathfrak{P}')])$$

is functorial. This proves the internal family reduction. □

Theorem 9.6 (Internal transport reduction)**.** *For every $\mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1$, the boundary-polarization, carrier-rigidity, family-holonomy, determinant-line, and positivity theorems reduce functorially to the TFPT transport package with admissible cusp set*

$$\left\{1, \frac{2}{3}, \frac{1}{3}\right\},$$

exact branch pole δ_{ph}, and the unique first-generation winding lock.

Proof sketch. Once the boundary winding, rigid carrier generator, and rigid family holonomy are fixed, the admissible cusp spectrum is no longer ambient input; it is read from the singlet-projected carrier spectrum. The algebraic transport-pole theorem, the exact C_6 kernel, and the transport length sum rule then fix the admissible cusp set, the pole δ_{ph}, and the first-generation winding pattern functorially. □

Theorem 9.7 (Internal glueing and invariant reconstruction)**.** *For every $\mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1$, the data extracted in Theorems 9.4 to 9.6 assemble into a TFPT pre-class object whose invariant tuple is the canonical TFPT tuple*

$$\mathcal{I}_\star = \big(X, Y, [u_\Sigma], X_f^\circ, c_1(L_2), c_1(L_3), [\nabla_F^\star], a_\Sigma, \Lambda_{\mathrm{IR}}, x_{\mathrm{cl}}^\star\big).$$

Proof sketch. The carrier, family, transport, determinant-line, gravity, cosmology, and vacuum closures determine exactly the displayed invariant tuple on the rigid branch. The internal reductions above therefore glue into one TFPT pre-class object with the same invariant content as the canonical branch. □

Theorem 9.8 (Reduction to the canonical rigid orbit)**.** *For every ambient object $\mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1$, the rigid internal reduction lands in the canonical rigid orbit:*

$$\overline{\mathfrak{R}}(\mathfrak{P}) \cong \mathfrak{T}_\star.$$

Equivalently, the induced map on isomorphism classes

$$\pi_0(\widehat{\mathrm{PhysAdm}}_1) \longrightarrow \pi_0(\mathbf{TFPT}^{\mathrm{rig}}), \qquad [\mathfrak{P}] \longmapsto [\overline{\mathfrak{R}}(\mathfrak{P})],$$

has singleton image

$$\pi_0(\mathbf{TFPT}^{\mathrm{rig}}) = \{[\mathfrak{T}_\star]\}.$$

If $u : \mathfrak{P}_1 \to \mathfrak{P}_2$ is an ambient morphism, then $\overline{\mathfrak{R}}(u)$ is the unique rigid intertwiner between the two reduced objects.

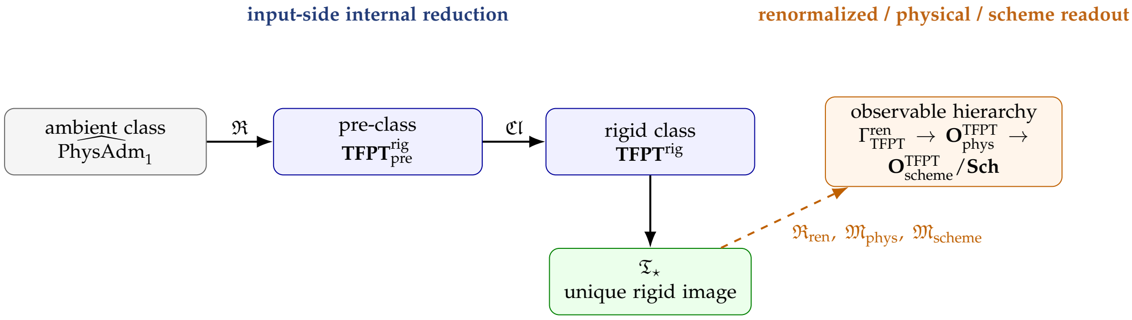


Figure 6. Input-side reduction and observable hierarchy.

Proof. Theorems 9.4 to 9.7 produce from every ambient object a TFPT pre-object with the canonical invariant tuple. By [TFPT cross-reference: `thm:branch-uniqueness`], that pre-object carries a unique intrinsic branch, and by [TFPT cross-reference: `thm:spectral-completeness-rigid-tfpt,cor:no-al` every rigid closed realization with this invariant tuple is uniquely isomorphic to $\mathfrak{T}_\star$. Therefore every reduced ambient object lies in the unique rigid orbit of $\mathfrak{T}_\star$. The functoriality on morphisms is immediate because the rigid target has only the unique intertwiner allowed by essential injectivity on the rigid image. □

Remark (Primitive honesty). The manuscript now proves the passage from a minimal operational seed to the one-sided boundary datum and then the canonical closure from that datum. It does not yet prove that $\mathbf{Seed}^{\mathrm{op}}$ itself is generated from a still thinner universal seed. Accordingly the manuscript establishes internal rigidity before ultimate primitive minimality.

Remark (Internal rigidity versus external adequacy). The main text proves uniqueness of the closed TFPT object inside the ambient class $\widehat{\mathrm{PhysAdm}}_1$:

$$\forall \mathfrak{P} \in \widehat{\mathrm{PhysAdm}}_1, \qquad \overline{\mathfrak{R}}(\mathfrak{P}) \cong \mathfrak{T}_\star.$$

This does not by itself prove that $\widehat{\mathrm{PhysAdm}}_1$ is the physically correct ambient class. External adequacy is a separate comparison claim on the observable map

$$\mathcal{E} : \widehat{\mathrm{PhysAdm}}_1 \to \mathcal{O}_{\mathrm{phys}}, \qquad \mathfrak{P} \longmapsto \mathcal{E}(\mathfrak{P}).$$

The input-side reduction and the output-side orbit are summarized in Figure 6.

9.2 Renormalized, physical, and scheme readout layers

Definition 9.9 (Admissible effective average action)**.** Let R_k be an admissible infrared regulator commuting with the structural sector selection. Define

$$e^{W_k[J]} := \int_{\mathcal{H}_{\mathrm{adm}}} \mathcal{D}\varphi \, \exp\Big(-S_{\mathrm{adm}}[\varphi] - \frac{1}{2}\langle \varphi, R_k \varphi\rangle + \langle J, \varphi\rangle\Big),$$

and set

$$\Gamma_k[\phi] := \sup_J \{\langle J, \phi\rangle - W_k[J]\} - \frac{1}{2}\langle \phi, R_k \phi\rangle.$$

The renormalized admissible effective action is

$$\Gamma^{\mathrm{ren}}_{\mathrm{TFPT}} := \lim_{k \to 0} \Gamma_k.$$

Theorem 9.10 (Exact admissible renormalization-group flow)**.** *The admissible effective average action satisfies the exact functional flow equation*

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{STr}_{\mathcal{H}_{\mathrm{adm}}}\Big[\big(\Gamma_k^{(2)}[\phi] + R_k\big)^{-1}\partial_t R_k\Big], \qquad t := \log k.$$

Thus the running couplings of the admissible theory are generated dynamically on Γ_k*, not by the static projector* P_{adm}*.*

Proof. Differentiate the Legendre transform defining Γ_k at fixed mean field ϕ. Using

$$\frac{\delta^2 W_k}{\delta J\,\delta J} = \big(\Gamma_k^{(2)} + R_k\big)^{-1},$$

one obtains

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{STr}\big[\big(\Gamma_k^{(2)} + R_k\big)^{-1}\partial_t R_k\big].$$

The supertrace is taken on the admissible physical sector because the regulated functional integral itself is defined there. □

Corollary 9.11 (One-loop shadow of the exact admissible flow)**.** *On the canonical branch with three families and one Higgs doublet, the exact admissible flow has the perturbative shadow*

$$\partial_t g_i^{-2}(k) = -\frac{b_i}{8\pi^2} + O(g_i^0), \qquad (b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7\right),$$

with g_1 *in the standard hypercharge normalization. Equivalently,*

$$\partial_t \alpha_1^{-1}(k) = -\frac{41}{20\pi} + O(\alpha_1).$$

Proof. Expanding Theorem 9.10 around the Gaussian region yields the standard one-loop coefficients determined by the field content of the canonical branch. The carrier and family theorems fix exactly three families and one light Higgs doublet, so the resulting shadow coefficients are the displayed Standard Model values. □

Corollary 9.12 (Infrared boundary condition for the electromagnetic coupling)**.** *Let* α_* *be the exact positive solution of the admissible electromagnetic closure equation. Then*

$$\alpha_{\mathrm{em}}(0) = \alpha_*$$

is the infrared boundary value of the exact admissible flow. For $\mu > 0$ *one has*

$$\alpha_{\mathrm{em}}^{-1}(\mu) = \alpha_*^{-1} - \int_0^{\log \mu} \beta_{\alpha^{-1}}(t)\,dt - \Delta_{\mathrm{thr}}(\mu),$$

where Δ_{thr} *records electroweak mixing and threshold decoupling.*

Proof. [TFPT cross-reference: `thm:alpha-canonical`] fixes the unique branch value α_* at the infrared endpoint. The exact flow of Theorem 9.10 then propagates that value to finite scales, while threshold effects are collected in the declared correction term Δ_{thr}. □

Theorem 9.13 (Admissible 1PI graph expansion)**.** *Around any admissible background field* ϕ*, the effective action admits the loop expansion*

$$\Gamma_k[\phi] = S_{\mathrm{adm}}[\phi] + \frac{\hbar}{2}\,\mathrm{STr}\log\big(S_{\mathrm{adm}}^{(2)}[\phi] + R_k\big) + \sum_{\ell \geq 2} \hbar^\ell \Gamma_{\ell,k}^{\mathrm{1PI}}[\phi].$$

Each coefficient $\Gamma^{1\mathrm{PI}}_{\ell,k}$ is the sum of admissible one-particle-irreducible Feynman graphs with propagator

$$G_{k,\phi} := \left(S^{(2)}_{\mathrm{adm}}[\phi] + R_k\right)^{-1}$$

and vertices given by the higher functional derivatives $S^{(n)}_{\mathrm{adm}}[\phi]$.

Proof. Set $\varphi = \phi + \sqrt{\hbar}\eta$ and expand

$$S_{\mathrm{adm}}[\phi + \sqrt{\hbar}\eta] = S_{\mathrm{adm}}[\phi] + \frac{\hbar}{2}\langle \eta, S^{(2)}_{\mathrm{adm}}[\phi]\eta\rangle + \sum_{n\geq 3} \frac{\hbar^{n/2}}{n!} S^{(n)}_{\mathrm{adm}}[\phi](\eta^n).$$

The regulated functional integral therefore becomes a Gaussian measure with covariance $G_{k,\phi}$ dressed by higher interaction vertices. Taking the logarithm produces the connected diagram expansion, and the Legendre transform removes one-particle-reducible pieces. The resulting coefficients are exactly the admissible 1PI graph sums recorded above. □

Definition 9.14 (Canonical renormalized 1PI package)**.** Let $\Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}$ denote the category whose objects are renormalized low-energy packages

$$\mathcal{G} = \left(\{\Sigma^{\mathrm{TFPT}}_f(p)\}_f,\, G^{\mathrm{TFPT}}_F,\, \mathcal{R}_{\mathrm{det}},\, \mathcal{B}_{\mathrm{cos}},\, \mathcal{A}_{\mathrm{pole}}\right)$$

built canonically from the rigid object $\mathfrak{T}_\star$. They are extracted from the infrared limit of the exact admissible flow,

$$\Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}[\Phi_c] := \lim_{k\to 0} \Gamma_k[\Phi_c],$$

equivalently the Legendre transform of the retained relative generating functional after removing the infrared regulator. Here $\mathcal{A}_{\mathrm{pole}}$ collects the dressed inverse propagators, $\mathcal{R}_{\mathrm{det}}$ the determinant-line response kernels, and $\mathcal{B}_{\mathrm{cos}}$ the FRW / reheating / Boltzmann interface data on the closed branch. Morphisms are RG-compatible field reparameterizations preserving the branch-fixed pole and response data.

Definition 9.15 (Physical observable category)**.** Let $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}$ denote the category whose objects are triples

$$\mathcal{O}_{\mathrm{phys}} = (v_{\mathrm{phys}}, \Sigma, \mathcal{D}),$$

where v_{phys} is a package of physical observables (pole masses, widths, mixing angles, response coefficients, and closed-branch cosmology interface packages before late-time continuation), Σ collects their covariance data, and $\mathcal{D}$ is a dependency DAG recording algebraic and statistical non-independence.

Definition 9.16 (Scheme groupoid)**.** Let **Sch** be the groupoid whose objects are declared comparison-scheme packages

$$s = (s_{\mathrm{EM}}, s_{\mathrm{EW}}, s_\nu, s_{\mathrm{QCD}}, s_{\mathrm{cos}}, \Sigma_{\mathrm{snap}})$$

consisting of threshold conventions, mass schemes, oscillation conventions, cosmology snapshot rules, and covariance bookkeeping. Morphisms

$$\psi : s \to s'$$

are finite renormalization, threshold-transfer, snapshot-transfer, and covariance-preserving scheme changes.

Definition 9.17 (Scheme observable category)**.** Let $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}$ denote the category whose objects are triples

$$\mathcal{O}_{\mathrm{scheme}} = (v, \Sigma, \mathcal{D}),$$

where v is a scheme-specified representative of a physical observable package, Σ its declared covariance data, and $\mathcal{D}$ the induced dependency DAG. Morphisms are smooth covariance-preserving reparameterizations respecting $\mathcal{D}$.

Definition 9.18 (Canonical readout hierarchy)**.** Define the functors

$$\mathfrak{R}_{\mathrm{ren}} : \mathbf{TFPT}^{\mathrm{rig}} \to \Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}, \qquad \mathfrak{M}_{\mathrm{phys}} : \Gamma^{\mathrm{ren}}_{\mathrm{TFPT}} \to \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}, \qquad \mathfrak{M}_{\mathrm{scheme}} : \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}} \to \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}/\mathbf{Sch}.$$

For every scheme object $s \in \mathbf{Sch}$, let

$$\mathfrak{M}^{(s)}_{\mathrm{scheme}}\big(\mathfrak{M}_{\mathrm{phys}}(\mathfrak{R}_{\mathrm{ren}}(\mathfrak{T}_\star))\big) \in \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}$$

be the chosen scheme representative. The canonical appendix comparison orbit is

$$[\mathfrak{M}(\mathfrak{T}_\star)]_{\mathbf{Sch}} := \mathfrak{M}_{\mathrm{scheme}}\big(\mathfrak{M}_{\mathrm{phys}}(\mathfrak{R}_{\mathrm{ren}}(\mathfrak{T}_\star))\big) \in \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}/\mathbf{Sch},$$

and the appendix comparison map is any chosen representative

$$\mathcal{R}^{(s)}_{\mathrm{cmp}}(\mathfrak{T}_\star) := \mathfrak{M}^{(s)}_{\mathrm{scheme}}\big(\mathfrak{M}_{\mathrm{phys}}(\mathfrak{R}_{\mathrm{ren}}(\mathfrak{T}_\star))\big).$$

Definition 9.19 (Minimal independent observable basis)**.** Define the minimal independent basis package

$$\mathcal{B}_{\mathrm{ind}}(\mathfrak{T}_\star) := \big(C_{\mathrm{em}}, K_{\mathrm{UV}}, F_{\mathrm{fl}}, N_\nu, R_{\mathrm{CS}}, T_{\mathrm{phys}}, C_{\mathrm{cos}}, C_{\mathrm{CP}}\big),$$

where

$$C_{\mathrm{em}} := \alpha^{-1}(0), \qquad K_{\mathrm{UV}} := \varphi_0^{\mathrm{ret}}, \qquad F_{\mathrm{fl}} := (\lambda_C, \delta_{\mathrm{CKM}}), \qquad R_{\mathrm{CS}} := \beta, \qquad C_{\mathrm{CP}} := \theta_{\mathrm{eff}},$$

$$N_\nu := (\sin^2\theta_{13}, \delta^{\nu}_{\mathrm{CP}}, \sin^2\theta_{23}, \Sigma m_\nu, m_{\beta\beta}, M_R),$$

$$T_{\mathrm{phys}} := \big(G_F^{\mathrm{TFPT}}, m_W^{\mathrm{pole}}, m_Z^{\mathrm{pole}}, m_H^{\mathrm{pole}}, \{m_f^{\mathrm{pole}}\}_f\big),$$

$$C_{\mathrm{cos}} := \big(\Omega_b, S_\Sigma, \Lambda_{\mathrm{IR}}, T_R, f_a, m_a, \nu_a, \eta_B\big).$$

The retained seed $K_{\mathrm{UV}} = \varphi_0^{\mathrm{ret}}$ is part of the UV kernel package; it is no longer the direct basis package of the flavor, neutrino, determinant-response, or cosmology observables.

Theorem 9.20 (Renormalized observable hierarchy from the admissible 1PI action)**.** *There exist functors*

$$\mathfrak{R}_{\mathrm{ren}} : \mathbf{TFPT}^{\mathrm{rig}} \to \Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}, \qquad \mathfrak{M}_{\mathrm{phys}} : \Gamma^{\mathrm{ren}}_{\mathrm{TFPT}} \to \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}, \qquad \mathfrak{M}_{\mathrm{scheme}} : \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}} \to \mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}/\mathbf{Sch}$$

with the following properties:

- ***(O1) Reality first.*** *Main-text observables factor through $\mathfrak{M}_{\mathrm{phys}} \circ \mathfrak{R}_{\mathrm{ren}}$ and therefore belong to $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}$ before any scheme choice is made.*
- ***(O2) Unique factorization through the basis.*** *Every physical readout row and every appendix scheme row factors through exactly one component package of $\mathcal{B}_{\mathrm{ind}}(\mathfrak{T}_\star)$ via a declared physical or scheme map.*
- ***(O3) Scheme covariance.*** *For every scheme morphism $\psi : s \to s'$ one has*

$$\psi\Big(\mathfrak{M}^{(s)}_{\mathrm{scheme}}\big(\mathfrak{M}_{\mathrm{phys}}(\mathfrak{R}_{\mathrm{ren}}(\mathfrak{T}_\star))\big)\Big) = \mathfrak{M}^{(s')}_{\mathrm{scheme}}\big(\mathfrak{M}_{\mathrm{phys}}(\mathfrak{R}_{\mathrm{ren}}(\mathfrak{T}_\star))\big).$$

(O4) ***No hidden continuous fit parameter.*** *Any continuous freedom in a readout row is either a declared scheme degree of freedom in* **Sch** *or already a theorem-level branch parameter fixed in* $\mathfrak{T}_\star$.

(O5) ***Complete kill surface.*** *Every admissible empirical kill test of the manuscript is the pullback of a component test on* $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}$ *and only afterwards, if needed, of a scheme representative in* $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}$.

Proof. Define $\mathfrak{R}_{\mathrm{ren}}$ by sending the rigid object $\mathfrak{T}_\star$ to the renormalized 1PI package extracted from the exact admissible flow of Theorem 9.10. Its infrared endpoint is

$$\Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}[\Phi_c] = \lim_{k\to 0} \Gamma_k[\Phi_c],$$

equivalently the Legendre transform of the retained admissible generating functional after removing the infrared regulator. By Theorem 9.13, the same object carries the admissible 1PI graph expansion around each admissible background. Reflection positivity, lower boundedness, and coercivity on the admissible sector ensure the convexity needed for this Legendre construction in a neighborhood of the closed branch. Hence the dressed inverse propagators, determinant-response kernels, and cosmology saddle system belong to one and the same renormalized package $\Gamma^{\mathrm{ren}}_{\mathrm{TFPT}}$.

Next define $\mathfrak{M}_{\mathrm{phys}}$ by reading physical observables from that 1PI package before any scheme choice: pole masses are the zeros of the dressed inverse propagators, mixing data are the branch invariants of the corresponding residue matrices, determinant-response observables are read from the exact response kernels, and cosmology rows factor through the same FRW / reheating / Boltzmann interface block. Final late-time numerical representatives are attached only after the declared comparison map. This proves **(O1)**.

For **(O2)**, the factorization through the minimal basis package $\mathcal{B}_{\mathrm{ind}}(\mathfrak{T}_\star)$ is exactly the theorem-level content of the rewritten sector split. The flavor rows factor through F_{fl}, neutrino rows through N_ν, determinant-response rows through R_{CS}, strong-CP rows through C_{CP}, cosmology rows through C_{cos}, and the retained seed φ_0^{ret} survives only in the UV kernel package K_{UV}. No row uses two basis packages at once.

For **(O3)**, every scheme morphism in **Sch** acts only after $\mathfrak{M}_{\mathrm{phys}}$ has produced a physical observable package. Therefore threshold transfer, finite renormalization, and snapshot change act on scheme representatives without altering the underlying physical point, exactly as in the displayed covariance identity.

For **(O4)**, every continuous parameter entering a readout row is already fixed either by the closed branch $\mathfrak{T}_\star$ or by the declared scheme object $s \in \mathbf{Sch}$. Since $\mathfrak{R}_{\mathrm{ren}}$ and $\mathfrak{M}_{\mathrm{phys}}$ are defined from the fixed admissible measure and its 1PI action, no extra continuous fit parameter can appear in between.

Finally, **(O5)** follows because every empirical kill criterion in the manuscript is attached to one of the physical rows in $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{phys}}$ and only afterwards, if needed, represented by a scheme object in $\mathbf{O}^{\mathrm{TFPT}}_{\mathrm{scheme}}/\mathbf{Sch}$. Thus the whole observable hierarchy factors through the same admissible 1PI action and then through the declared physical / scheme split. □

Corollary 9.21 (No double counting after the seed-shadow split)**.** *Rows sharing the same basis package in Definition 9.19 define one underlying operational direction, not several independent confirmations or failures. In particular, the former quartet*

$$(\beta, \Omega_b, \lambda_C, \sin^2\theta_{13})$$

no longer constitutes one seed package. Instead

$$\beta \in R_{\mathrm{CS}}, \qquad \Omega_b \in C_{\mathrm{cos}}, \qquad \lambda_C \in F_{\mathrm{fl}}, \qquad \sin^2\theta_{13} \in N_\nu,$$

while the retained seed φ_0^{ret} *belongs only to the UV kernel package* K_{UV}.

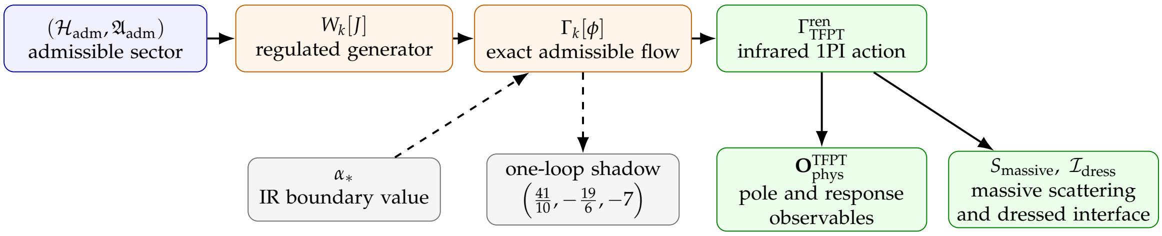


Figure 7. Exact admissible flow from the physical sector to observables and scattering. $P_{\rm adm}$ selects states; Γ_k carries running couplings and 1PI graphs.

Proof. This is the special case of **(O2)** for the rewritten basis packages of Definition 9.19. □

Corollary 9.22 (Independent response-sector birefringence target). *On the minimal branch the determinant-response package $R_{\rm CS}$ fixes*

$$\beta_{\rm rad} = \frac{\varphi^0_{\rm ret}}{4\pi},$$

and hence the dimensionless birefringence readout itself. Under the declared comparison interface it is represented by

$$\beta = \frac{180}{\pi}\,\beta_{\rm rad} = 0.2424350309009295\ldots^\circ.$$

This target factors through $R_{\rm CS}$ and not through the flavor package $F_{\rm fl}$ or the cosmology package $C_{\rm cos}$.

Proof. By the exact seed relation of the determinant-response channel,

$$\beta_{\rm rad} = \frac{\varphi^0_{\rm ret}}{4\pi}.$$

The theorem-level quantity is therefore the dimensionless angle $\beta_{\rm rad}$. Converting to degrees gives one scheme-facing representative,

$$\beta = \frac{180}{\pi}\beta_{\rm rad} = \frac{180}{\pi}\frac{\varphi^0_{\rm ret}}{4\pi} = 0.2424350309009295\ldots^\circ.$$

By [TFPT cross-reference: `cor:no-double-counting`], β belongs to $R_{\rm CS}$, whereas $\lambda_C \in F_{\rm fl}$ and $\Omega_b \in C_{\rm cos}$. Hence this is an operational direction distinct from the flavor and cosmology packages. □

Corollary 9.23 (Independent cosmology-sector axion target). *On the closed cosmology branch the package $C_{\rm cos}$ fixes the axion readout as one dimensionless cosmology package, equivalently as the boundary-normalized row*

$$\left(\frac{f_a}{\lambda_\Sigma}, \frac{m_a}{\lambda_\Sigma}, \frac{\nu_a}{\lambda_\Sigma}, \lambda_\Sigma \left|g^{\rm phys}_{a\gamma\gamma}\right|\right),$$

This target factors through $C_{\rm cos}$ and not through the flavor package $F_{\rm fl}$ or the determinant-response package $R_{\rm CS}$.

Proof. The closed cosmology branch fixes the determinant-line / seam-transfer / scalaron saddle and thus the axion row as a single component of $C_{\rm cos}$. Its unitful representatives belong only to the declared comparison interface and are recorded in the appendix benchmark layer, together with the practical haloscope window and coupling representative extracted from the same cosmology package. By [TFPT cross-reference: `cor:no-double-counting`], this row belongs to $C_{\rm cos}$, whereas β belongs to $R_{\rm CS}$ and λ_C belongs to $F_{\rm fl}$. Hence the axion target is an operational direction distinct from response and flavor. □

Remark (Why these two targets count as independent). The pair

$$\beta \in R_{\mathrm{CS}}, \qquad (\nu_a, m_a, g^{\mathrm{phys}}_{a\gamma\gamma}) \in C_{\mathrm{cos}},$$

samples two different basis packages of $\mathcal{B}_{\mathrm{ind}}(\mathfrak{T}_\star)$. Therefore they are not two rewritings of the same seed constraint. This is the correct sharp-prediction surface for the present version.

10 Relative reflection positivity with fermions

10.1 Normalized quotient form

The relative Euclidean theory used in the main closure section is not interpreted as a difference of measures. Instead it is organized as the normalized quotient

$$Z_{\mathrm{rel}}[J, \eta, \bar{\eta}] = \frac{Z_{\mathrm{adm}}[J, \eta, \bar{\eta}]}{Z_{\mathrm{ref}}[0, 0, 0]}.$$

Because the denominator is source independent on the admissible sector, the reference theory acts only as a fixed normalizer and does not interfere with reflection positivity.

10.2 Bosonic and fermionic positivity on P_{adm}

The commutation relations

$$[\Theta, P_\Theta] = 0, \qquad \Theta P_{\mathrm{adm}} = P_{\mathrm{adm}}\Theta$$

ensure that reflection preserves the admissible subspace. The bosonic part then follows the standard Markov / reflection-stable Osterwalder–Schrader argument on admissible observables. For fermions, the admissible CAR kernel may be written as a reflected quadratic form; equivalently one may express the same condition as positivity of the associated Pfaffian structure on the admissible sector. Hence for every admissible observable O supported in $\tau > 0$,

$$\langle \Theta O, O \rangle_{\mathrm{rel}} = \frac{1}{Z_{\mathrm{ref}}[0, 0, 0]} \langle \Theta O, O \rangle_{\mathrm{adm}} \geq 0.$$

This appendix section is the mechanism behind [TFPT cross-reference: `lem:admissible-reflection-positivi` it is recorded here so that the main-text reconstruction theorem is not supported only by a one-line axiom call.

11 Admissible OS reconstruction and asymptotic scattering

This appendix upgrades the admissible Schwinger, reconstruction, and scattering block from a named standard interface to a manuscript-level proof package. The point is not to replace standard machinery by ad hoc notation, but to verify explicitly that the admissible branch satisfies the relevant hypotheses on P_{adm}.

11.1 Tempered admissible Schwinger families

Lemma 11.1 (Temperedness and moment bounds for admissible Schwinger functions)**.** *On the Euclidean continuation of $\mathcal{U}_{\mathrm{scat}}$, each truncated admissible Schwinger distribution S_n^T defines a tempered distribution. Moreover, for every compact K and every source monomial of total degree n, its moments obey polynomial bounds determined by the same uniform heat-kernel estimates and finite local moments of the admissible constructive measure.*

Proof. By [TFPT cross-reference: `thm:well-posed-primitive-dynamics`], the admissible Euclidean generator is realized by a self-adjoint elliptic operator on every blocked volume. Together with the constructive measure of [TFPT cross-reference: `thm:constructive-geometric-measure`], this yields uniform heat-kernel bounds and finite moments for local source insertions. Differentiating W_{rel} with respect to the local sources therefore produces distributions whose derivatives are controlled by polynomial combinations of the same bounds. Passing to the admissible projective limit preserves these polynomial estimates, so the resulting Schwinger family is tempered. □

Lemma 11.2 (Graded Euclidean covariance and symmetry on P_{adm})**.** *On the Euclidean continuation of $\mathcal{U}_{\text{scat}}$, the admissible Schwinger family is graded Euclidean covariant, graded symmetric under exchange of external legs, and continuous in the Schwartz topology.*

Proof. The Dirac-type principal symbol of the geometric anchor and the low-curvature reduction imply local Euclidean covariance of the admissible source functional on $\mathcal{U}_{\text{scat}}$. Bosonic source derivatives commute, while fermionic derivatives anticommute, so the mixed Schwinger family is graded symmetric with the usual CAR signs. Continuity follows from the tempered moment bounds of Lemma 11.1, which control the action of S_n^T on Schwartz test functions. □

11.2 OS and CAR reconstruction on P_{adm}

Theorem 11.3 (Tailored OS/CAR reconstruction on P_{adm})**.** *Let $\{S_n^T\}_{n\geq 1}$ be the admissible Schwinger family on the Euclidean continuation of $\mathcal{U}_{\text{scat}}$. If reflection positivity, temperedness, graded Euclidean covariance, graded symmetry, and clustering hold on P_{adm}, then there exist:*

(i) a Hilbert space $\mathcal{H}_{\text{OS}}$ with vacuum vector Ω,

(ii) a strongly continuous positive-energy semigroup $T_t = e^{-tH_{\text{adm}}}$ with self-adjoint generator $H_{\text{adm}} \geq 0$,

(iii) a reconstructed graded local field net on P_{adm} whose Euclidean boundary values are the original admissible Schwinger functions.

Equivalently, the admissible Euclidean family reconstructs a Wightman/CAR theory on the admissible sector.

Proof. Define the OS sesquilinear form by

$$(F, G)_{\text{OS}} := \langle \Theta F, G \rangle_{\text{rel}}$$

for positive-time admissible observables. Reflection positivity from [TFPT cross-reference: `thm:reflection-positivity-admissible`] and the fermionic quotient-form analysis of Section 10 show that this form is positive semidefinite on the full graded admissible algebra. Quotienting by the null space and completing gives $\mathcal{H}_{\text{OS}}$.

Positive Euclidean time translation preserves the null space and acts by contractions, hence defines a strongly continuous contraction semigroup T_t on $\mathcal{H}_{\text{OS}}$. The dense domain of positive-time polynomial observables gives the symmetric generator domain. By Hille–Yosida, the closure of this generator is self-adjoint and nonnegative, so $T_t = e^{-tH_{\text{adm}}}$ for a self-adjoint $H_{\text{adm}} \geq 0$.

The temperedness, covariance, graded symmetry, and clustering established in Lemmas 11.1 and 11.2 are precisely the remaining OS/CAR reconstruction hypotheses. Therefore analytic continuation yields the reconstructed Wightman/CAR family with vacuum Ω. Because Θ, P_Θ, and P_{adm} commute, the entire reconstruction stays on the admissible sector P_{adm}. □

11.3 Stable massive Haag–Ruelle and LSZ closure

Lemma 11.4 (One-particle spectral projectors from isolated simple poles)**.** *Let $\widetilde{G}_a^{(2)}(p)$ have an isolated simple pole with positive residue at $p^2 = -m_a^2 < 0$. Then the corresponding one-particle contribution defines a positive spectral projector on the admissible Hilbert space.*

Proof. The isolated simple pole separates a one-particle mass shell from the multiparticle continuum. By positivity of the residue, the contour integral of the resolvent around that pole defines a positive projector on $\mathcal{H}_{\mathrm{adm}}$. Its range is the one-particle subspace associated with the stable species a. □

Theorem 11.5 (Haag–Ruelle scattering on the stable massive admissible sector)**.** *For every stable massive admissible species with isolated simple pole below threshold, the Haag–Ruelle asymptotic fields exist on* $\mathcal{H}_{\mathrm{adm}}$ *and generate wave operators on the stable massive scattering sector.*

Proof. By [TFPT cross-reference: `thm:tailored-os-car-reconstruction,thm:admissible-local-minkowski-` the admissible branch carries a local relativistic net with positive Hamiltonian. By [TFPT cross-reference: `thm:nonperturbative-admissible-qft`], connected correlators cluster exponentially on the massive sector. The pole assumption and Lemma 11.4 identify the one-particle subspace and give positive single-particle spectral projectors. The standard Haag–Ruelle estimates therefore apply to admissible local operators, yielding the in/out fields and the wave operators on the stable massive sector. □

Corollary 11.6 (LSZ reduction on the stable massive admissible sector)**.** *On the stable massive admissible sector, matrix elements of the scattering operator are obtained by amputating the connected correlators with the corresponding residue factors.*

Proof. By Theorem 11.5, the asymptotic fields and wave operators exist on the stable massive sector. Applying the usual LSZ reduction to those asymptotic fields and the admissible local net yields the displayed amputated-correlator formula. □

11.4 Dressed massless interface in the low-curvature region

The massless sector is not treated by bare Fock asymptotics in the present manuscript. The relevant object is the infrared-dressed interface package

$$\mathcal{I}_{\mathrm{dress}} := (\Omega_+^{\mathrm{dress}}, \Omega_-^{\mathrm{dress}}, S_{\mathrm{dress}}), \qquad S_{\mathrm{dress}} := (\Omega_+^{\mathrm{dress}})^* \Omega_-^{\mathrm{dress}},$$

constructed from the soft charges of the admissible local net on the low-curvature region $\mathcal{U}_{\mathrm{scat}}$. This is exactly the interface package used in the main-text theorem [TFPT cross-reference: `thm:dressed-massless-interface`].

12 Yukawa kernels and positivity lemmas

This appendix is the proof motor for the transport positivity, determinant-phase suppression, and bounded-residual statements used in the main text. The transport program is based on the regularized kernel

$$\mathcal{Y}_f^{(\varepsilon)} = (D_f^\dagger D_f + \varepsilon^2)^{-1}.$$

Three remarks are important.

1. The regulator ε is not optional: without it the kernel becomes ill-defined exactly where the admissibility question is most delicate.
2. Positivity of the kernel belongs to the admissibility-closure layer rather than to the hard carrier kernel. The main text closes it through the finite hexagon gap and the lift to the full admissibility-projected operator; this appendix records the transport-side mechanism behind that closure.
3. Residual prefactors $\Lambda_{f,j}$ are bounded holonomy data inside the transport theorem. They must remain visible and must not be rhetorically hidden as if the mass map were diagonal without residual transport structure.

Remark (Positive-kernel determinant route to strong CP). If the physical mass map is written in the polar form

$$M_f = U_{f,L} H_f U_{f,R}^\dagger, \qquad H_f > 0, \qquad U_{f,L}, U_{f,R} \in SU(3)_F,$$

with H_f induced by the positive transport kernel $\mathcal{Y}_f^{(\varepsilon)}$, then the induced determinant phase vanishes:

$$\arg\det M_f = 0.$$

At the same time weak CP may survive through

$$V_{\rm CKM} = U_{u,L}^\dagger U_{d,L}.$$

Combined with the sheet-CP symmetry protection of $\theta_{\rm sheet}$ in the relative action, this is the algebraic core of the determinant-phase theorem stated in the main text and of the strong-CP closure theorem recorded there. The role of the appendix remark is to display the polar-decomposition mechanics behind that route, not to reopen its status.

12.1 Optional strong-coupling sharpening

As an appendix-level dynamical sharpening of the hadronic admissibility theorem, one may further require the admissible center-neutral Wilson functional to obey strong-coupling dominance. On that optional route the relative Wilson loop satisfies

$$\langle W(C)\rangle_{\rm rel} \le \exp[-\sigma_{\rm QCD} A_{\rm min}(C)], \qquad \sigma_{\rm QCD} = c_3^2 \chi_{\rm QCD}^2,$$

and the physical Hilbert space contains no color-nonsinglet asymptotic states. This statement is kept in the appendix so that the main theorem surface does not depend on an extra strong-coupling sharpening beyond the closed admissible hadronic sector.

13 Source Extraction Map

Source extraction map

Use `../tfpt-42.tex`:

- Sections 7.4 and 7.5 for hadronic admissibility and strong-CP ingredients.
- Section 8 as the main technical body, with carrier repetition removed.
- Sections 8.20–8.22 for topological positivity and strong-CP closure.
- Sections 8.23–8.41 for reflection positivity, OS closure, local net, and scattering.
- Section 13.2 for exact admissible RG flow, 1PI expansion, and observable hierarchy.
- Appendices E, F, and H as the technical backend.

Exported objects

Exports: $P_{\rm adm} = P_{\rm prim} P_{\rm sing} P_\Theta$, $\theta_{\rm eff} = 0$, admissible Schwinger/OS interface, local-net and RG-flow hypotheses/results under conditional closure.

14 Not Used Here

Exact electromagnetic benchmark derivations, CKM/PMNS prediction tables, gravity/metrology readouts, CMB spectra, sky realization, E8 grammar, full empirical ledgers, horizons, and transient channels are not used as proof inputs in this paper.