The Gaussian code bridge: E_8 over $\mathbb{Z}[i]$, the extended Hamming code, and a four-bit information layer

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Abstract

Build the E_8 lattice by Construction A over the extended Hamming code $[8, 4, 4]$, placed equivariantly with respect to four fixed coordinate pairs, and let J be the complex structure that rotates each pair ($J^2 = -1$, $\mu_4 = \langle J \rangle$). Then L becomes a unimodular Hermitian $\mathbb{Z}[i]$-lattice of rank 4, and reduction modulo the ramified Gaussian prime $1 + i$ produces a canonical four-bit quotient $L/(1 + i)L \cong \mathbb{F}_2^4$. We prove: the 240 roots avoid the zero class — by the two-line norm argument $|(1+J)x|^2 = 2|x|^2$ against the doubly-even minimum 4 — and distribute exactly 15×16 over the fifteen nonzero classes, each class being a union of exactly 4 of the 60 Gaussian root lines. On this quotient the pair 3-cycle σ (order 3, commuting with J) acts as a family 3-cycle with a fixed anchor bit — the information-bit action of the Reed–Muller code RM$(1, 3)$ — with residual identification gauge of order exactly 18, and the sixteen coordinate roots $\pm 2e_i$ form precisely one class, the σ-fixed label $F_1+F_2+F_3$. The classes cut across the Construction-A codeword fibers: the code that builds the lattice returns, after Gaussian reduction, as its *message space*. A quartic companion theorem types the invariant-theoretic shadow: the restrictions of the Weyl-invariant root power sums $F_8, F_{12}, F_{20}, F_{24}$ to the holomorphic eigenspace $\ker(J - i)$ are algebraically independent invariants of the complex reflection group G_{31} and hence, by Chevalley's theorem, a system of basic invariants — while the vanishing of the complementary degrees $\{2, 14, 18, 30\}$ is proved but honestly typed as the trivial μ_4-orbit factor $1 + (-i)^d + (-1)^d + i^d = 0$ ($d \not\equiv 0 \bmod 4$), with no G_{31} content. Every statement is certified by two exact-arithmetic probes (26/26 and 22/22 checks, no floats) and 65 kernel-checked Lean 4 theorems, with must-fail controls (non-equivariant placements, non-integral complex structures, $\mathbb{Z}[i]^4$) that kill or trivialize the structure exactly as demanded. No claim beyond the stated algebra is made.

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1 The three objects

1.1 The code and its equivariant placement

Let $C \subset \mathbb{F}_2^8$ be the extended Hamming code $[8, 4, 4]$ — the unique doubly-even self-dual binary code of length 8, permutation equivalent to the first-order Reed–Muller code RM$(1, 3)$ [2, Ch. 1,

*Working note of the verification program of Topological Fixed-Point Theory (TFPT), https://fixpoint-theory.com. Every number quoted in this note is produced by a machine-checked artefact named in the text: the exact-arithmetic discovery probes `experiments/tfpt-discovery/gaussian_code_bridge_probe.py` (26/26 checks) and `quartic_half_probe.py` (22/22 checks), promoted to the permanent verification suite as the modules `v689`/`v690`, and the Lean 4 proof modules `TfptCarrier/GaussianCodeBridge.lean` and `TfptCarrier/QuarticHalf.lean` ($44 + 21 = 65$ kernel-checked theorems, no `sorry`, no `native_decide`). No number is quoted that was not printed by an executed run or certified by the Lean kernel.

13–14]. Its automorphism group has order $|\mathrm{AGL}(3,2)| = 1344$, so C admits exactly $8!/1344 = 30$ distinct placements (coordinate images) inside $\mathbb{F}_2^8$. Fix the four μ_4 *pairs* $(0,1),(2,3),(4,5),(6,7)$ and the two permutations

$$\pi_J = (01)(23)(45)(67) \quad \text{(in-pair swap)}, \qquad \pi_\sigma : \ (0,1) \to (2,3) \to (4,5) \to (0,1), \ (6,7) \text{ fixed}$$

(the pair 3-cycle). A machine census of all 30 placements shows that *exactly two* are invariant under both π_J and π_σ (probe check I0.1); we call C^* the one containing the indicator vector of $\{0,2,4,6\}$, and note that the other is its image under the single transposition (67). The fourteen weight-4 supports of C^* have a closed description: the six *pair unions* (three family–family: $\{0,1,2,3\},\{0,1,4,5\},\{2,3,4,5\}$; three family–anchor: $\{0,1,6,7\},\{2,3,6,7\},\{4,5,6,7\}$) and the eight *even transversals* $t(d) = \{2k + d_k : k = 0,\dots,3\}$ with $d \in \mathbb{F}_2^4$, $\sum_k d_k$ even — one coordinate from each pair.

1.2 The lattice

Let $L = A(C^*) = \{x \in \mathbb{Z}^8 : \ x \bmod 2 \in C^*\}$ be the Construction-A lattice of C^* [1, Ch. 5], with the standard inner product $\langle x, y\rangle$ of $\mathbb{Z}^8$. Then $[\mathbb{Z}^8 : L] = 2^4 = 16$, and since C^* is doubly even, writing $x = c + 2m$ with $c \in \{0,1\}^8$ a codeword lift gives

$$\langle x, x\rangle = \mathrm{wt}(c) + 4(c \cdot m + m \cdot m) \equiv \mathrm{wt}(c) \equiv 0 \pmod 4, \tag{1}$$

so every nonzero $x \in L$ has $\langle x, x\rangle \geq 4$: the minimum of L is 4, attained by exactly 240 vectors (the 16 coordinate vectors $\pm 2e_i$ and the 224 lifts $c + 2m$ of weight-4 codewords with support-confined sign choices). Rescaled by $1/\sqrt{2}$, L is a copy of the even unimodular root lattice E_8; we work integrally in L throughout and call the 240 minimal vectors the *roots*.

1.3 The complex structure and the clock

Let $J \in O_8(\mathbb{Z})$ rotate each μ_4 pair, $e_{2k} \mapsto e_{2k+1} \mapsto -e_{2k}$, and let $\sigma \in O_8(\mathbb{Z})$ be the pair 3-cycle $e_0 \mapsto e_2 \mapsto e_4 \mapsto e_0$, $e_1 \mapsto e_3 \mapsto e_5 \mapsto e_1$, fixing e_6, e_7. Then (machine checks I0.2–I0.3; Lean theorems `J_sq`, `J_skew`, `sigma_cubed_ambient`, `J_sigma_commute`, `lattice_J`, `lattice_sigma`)

$$J^2 = -1, \qquad J^{\mathsf{T}} = -J, \qquad \sigma^3 = 1, \qquad [J,\sigma] = 0, \qquad JL = L, \qquad \sigma L = L,$$

where the last two use precisely the π_J- and π_σ-invariance of the placement C^* (sign changes are invisible mod 2). Both J and σ preserve the root set, so both lie in the Weyl group $W(E_8)$ (E_8 has no diagram automorphisms). Setting $i \cdot x := Jx$ makes L a torsion-free $\mathbb{Z}[i]$-module of rank 4, and σ is $\mathbb{Z}[i]$-linear. In the source modules of the verification program, σ arises as the fourth power of an order-12 "clock" element; here it is simply the explicit matrix above.

Lemma 1 (Hermitian unimodularity)**.** *Define $h(x,y) := \frac{1}{2}\big(\langle x, y\rangle + i\,\langle x, Jy\rangle\big)$. Then h is a $\mathbb{Z}[i]$-valued Hermitian form on L with $h(x,x) = \frac{1}{2}\langle x, x\rangle$, minimum 2, and (L,h) is* unimodular*: the Hermitian dual of L is L itself. Moreover $|(1+J)x|^2 = 2\,|x|^2$ for all x, i.e. multiplication by $1+i$ doubles norms exactly.*

Proof. By (1) and polarization, $\langle x, y\rangle \in 2\mathbb{Z}$ for all $x, y \in L$; since $JL = L$, also $\langle x, Jy\rangle \in 2\mathbb{Z}$, so h takes values in $\mathbb{Z}[i]$. Sesquilinearity is the two identities $\langle Jx, y\rangle = -\langle x, Jy\rangle$ (skewness) and $\langle Jx, Jy\rangle = \langle x, y\rangle$ (orthogonality), which give $h(Jx, y) = i\,h(x,y)$; skewness also gives $\langle x, Jx\rangle = 0$, hence $h(x,x) = \frac{1}{2}\langle x, x\rangle \geq 2$. For unimodularity: the $\mathbb{Z}$-dual of L is $L^\sharp = \frac{1}{2}L$ (determinant $16^2 = 256$), and $h(x, L) \subseteq \mathbb{Z}[i]$ is equivalent, using $JL = L$, to $\langle x, L\rangle \subseteq 2\mathbb{Z}$, i.e. $x \in 2L^\sharp = L$. Finally $|(1+J)x|^2 = |x|^2 + 2\langle x, Jx\rangle + |Jx|^2 = 2|x|^2$; in matrix form $(1+J)^{\mathsf{T}}(1+J) = 2 \cdot \mathbf{1}$, certified by the Lean kernel (theorem `one_add_J_gram`). □

2 The four-bit quotient and the 15×16 census

All checks of this section are from `gaussian_code_bridge_probe.py` (26 checks, 0 failures, verdict GAUSSIAN-CODE-BRIDGE-EXACT; exact integer and `Fraction` arithmetic throughout, `sympy` only for the Smith normal form), with the finite certificates re-proved by the Lean kernel in `GaussianCodeBridge.lean`.

Theorem 2 (the Gaussian code bridge, part I: the census)**.** *Let $L = A(C^*)$ with the $\mathbb{Z}[i]$-structure of Section 1, and let $(1+i)L = (1+J)L$.*

(i) $2L \subseteq (1+i)L$*, the index is* $[L : (1+i)L] = \det(1+J) = 2^4 = 16 = N(1+i)^4$*, and*

$$L/(1+i)L \;\cong\; \mathbb{F}_2^4$$

as an $\mathbb{F}_2$-vector space; the elementary divisors of $(1+i)L \subseteq L$ are $(1,1,1,1,2,2,2,2)$.

(ii) ***The zero class is empty on roots****: no root lies in $(1+i)L$. Indeed $\min L = 4$ by (1), and by Lemma 1 $\min (1+i)L = 2 \cdot 4 = 8 > 4$.*

(iii) ***Equidistribution****: each of the 15 nonzero classes contains exactly 16 roots, $240 = 15 \times 16$ (finite kernel certificate).*

(iv) ***μ_4-stability****: $Jx \equiv x \bmod (1+i)L$ for every $x \in L$; in particular every class is μ_4-stable.*

(v) J acts freely on the roots, partitioning them into 60 lines $\{\pm x, \pm Jx\}$ (the Gaussian root lines); each nonzero class is a union of exactly 4 lines, $60 = 15 \times 4$.

Proof. (i) $2 = -i\,(1+i)^2$ in $\mathbb{Z}[i]$, so $2L \subseteq (1+i)L$ and the quotient is elementary 2-abelian. On each pair, $1+J$ acts by $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ of determinant 2, so $\det(1+J) = 2^4 = 16$ and the quotient has order 16, hence $\mathbb{F}_2$-dimension 4. The elementary divisors are certified in Lean by an explicit Smith certificate: integer matrices P, Q, unimodular in both directions ($PP^{-1} = P^{-1}P = QQ^{-1} = Q^{-1}Q = \mathbf{1}$), with $PAQ = \mathrm{diag}(1,1,1,1,2,2,2,2)$ and the converse ingredient $DQ^{-1} = PA$, where A expresses $(1+i)b_j$ in a $\mathbb{Z}$-basis (b_j) of L (theorems `smith_certificate`, `P_unimodular`, `Q_unimodular`, `smith_converse_ingredient`, `index_sixteen`, `snf_exponent_two`).

(ii) is the displayed two-line argument; the probe additionally enumerates all vectors of norm < 4 (none) and finds 0 roots in the zero class (check I2.2); the Lean census re-proves it exactly (`census_zero_class_empty`).

(iii) is a finite decision: the probe reduces all 240 roots (check I2.3, census $\{16\colon 15\}$), and the Lean kernel certifies, over the explicit list of the 240 root coordinate vectors (pairwise distinct, norm 4, congruent to codewords mod 2), that the SNF residue map hits every nonzero label exactly 16 times (`census_equidistribution`).

(iv) $i - 1 = i(1+i)$, so $(J-1)x = J(1+J)x \in (1+i)L$ for all $x \in L$. (Lean: the μ_4 transport acts as the identity on the label block, `mu4_acts_trivially_on_labels`, and relabeling all 240 roots after the J-action reproduces the label list verbatim, `census_mu4_stable`.)

(v) $Jx = \pm x$ forces $x = 0$ (apply J once more and use $J^2 = -1$), so the μ_4-orbits $\{\pm x, \pm Jx\}$ of roots have size 4 and there are $240/4 = 60$ of them (check I2.5). Given (iii) and (iv), each class contains $16/4 = 4$ complete lines. □

Remark 3 (what needs the placement and what does not). Parts (i)–(v) use only the π_J-invariance of the placement: the probe re-runs the census over the *non-equivariant* naive placement (which is π_J- but not π_σ-invariant) and finds the same 15×16 equidistribution and 4-lines-per-class structure (check I4.2). The census is a property of the pair (E_8, J) alone. What dies without π_σ-equivariance is exactly the semantics of Section 3 (control I4.1). The census is also presentation independent: re-run in the standard E_8 coordinates (doubled model, basis determinant $256 = 2^8$), it returns the same Smith form and the same 15×16 census (check I5.1).

3 The information layer

Since σ is $\mathbb{Z}[i]$-linear and preserves L, it preserves $(1+i)L$ and descends to an $\mathbb{F}_2$-linear map $\bar{\sigma}$ on $L/(1+i)L \cong \mathbb{F}_2^4$. (Well-definedness is itself a kernel certificate: the transport matrix of σ into Smith coordinates has even top-right block, Lean theorem `sigma_label_block`.)

Theorem 4 (the Gaussian code bridge, part II: the information layer)**.** *(i)* $\bar{\sigma}^3 = 1$, $\bar{\sigma} \neq 1$, *and the fixed space of* $\bar{\sigma}$ *has dimension* 2 *(exactly* 4 *fixed labels,* 3 *of them nonzero).*

(ii) On the 15 *root-bearing classes,* $\bar{\sigma}$ *acts with orbit census* 3 *fixed classes* + 4 *three-cycles.*

(iii) There is a basis (F_1, F_2, F_3, A) *of* $\mathbb{F}_2^4$ *with*

$$\bar{\sigma} : \; F_1 \to F_2 \to F_3 \to F_1, \qquad \bar{\sigma} A = A :$$

$\bar{\sigma}$ *is the* family 3-cycle with fixed anchor — *the information-bit action of the equivariant* $\mathrm{RM}(1,3)$ *placement, in which each* μ_4 *pair carries one message bit, the three moving pairs are the families and the fixed pair is the anchor. The identification with the* $\mathrm{RM}(1,3)$ *message space is unique up to the centralizer of* $\bar{\sigma}$ *in* $\mathrm{GL}(4,\mathbb{F}_2)$*, which has order exactly* $18 = |\,\mathrm{GL}_2(\mathbb{F}_2)| \cdot |\mathbb{F}_4^\times| = 6 \times 3$.

(iv) ***The coordinate-class law****: the* 16 *coordinate roots* $\pm 2e_i$ *form exactly one class, and that class contains nothing else; it is* σ*-fixed and equals* $F_1 + F_2 + F_3$*. Thus* $240 = 14 \times 16 + 16$ *with the coordinate block as the fifteenth message label, the "sum of the families".*

Proof. (i) and (iv) are finite kernel certificates (Lean theorems `sigma_cubed`, `sigma_nontrivial`, `sigma_fixed_space`, `sigma_fixed_nontrivial`, `family_three_cycle`, `anchor_fixed`, `family_anchor_spans`, `coord_root_ambient`, `coord_class_eq_family_sum`, `coord_class_sigma_fixed`; probe checks I2.6a–d, I3.1–2), including the direct census equivariance: relabeling all 240 roots after the σ-action equals the $M_{\bar{\sigma}}$-image of the label list (`census_sigma_equivariant`). (ii) is probe check I2.6c (orbit census $\{1\colon 3,\ 3\colon 4\}$). For the gauge count in (iii): since $|\langle\bar{\sigma}\rangle| = 3$ is odd, the $\mathbb{F}_2\langle\bar{\sigma}\rangle$-module $\mathbb{F}_2^4$ is semisimple and decomposes as $(\text{trivial})^2 \oplus \mathbb{F}_4$, so the centralizer is $\mathrm{GL}_2(\mathbb{F}_2) \times \mathbb{F}_4^\times$ of order $6 \times 3 = 18$; the probe confirms by enumerating all of $\mathrm{GL}(4,\mathbb{F}_2)$ (check I2.6e). The $\mathrm{RM}(1,3)$ reading of (iii) is the content of the equivariant placement: π_σ permutes the three family pairs cyclically and fixes the anchor pair, and $\bar{\sigma}$ realizes exactly this permutation matrix on the message space. (This last identification, up to the order-18 gauge, is verified numerically in the probe; the Lean layer formalizes (i), (ii) — via the fixed-label count —, (iv) and the transport certificates, but not the gauge statement: see the honest scope in Section 5.) □

Remark 5 (the bridge is not the mod-2 reduction). The Gaussian classes cut *across* the Construction-A fibers (probe check I3.3): every class mixes codeword fibers, as the following complete census shows (probe I3.2 table; each fiber entry means $8+8$ roots over the two named weight-4 codewords, except the coordinate class). Here $t(d)$ is the even transversal with in-pair digits d, and $\bar{d}$ its complement.

classes (basis F_1, F_2, F_3, A)	$\bar{\sigma}$-action	codeword fibers of the 16 roots
$F_1,\ F_2,\ F_3$	3-cycle	8 over a family–anchor + 8 over a family–family word
$F_1{+}F_2,\ F_2{+}F_3,\ F_3{+}F_1$	3-cycle	8 over a family–anchor + 8 over a family–family word
$F_1{+}A,\ F_2{+}A,\ F_3{+}A$	3-cycle	$8+8$ over a complementary pair $t(d), t(\bar{d})$
$F_i{+}F_j{+}A$ (three classes)	3-cycle	$8+8$ over a complementary pair $t(d), t(\bar{d})$
A and $F_1{+}F_2{+}F_3{+}A$	fixed	$8+8$ over $t(0000)$ and $t(1111)$
$F_1{+}F_2{+}F_3$ (coordinate class)	fixed	all 16 over the zero codeword: the roots $\pm 2e_i$

The three $\bar{\sigma}$-fixed nonzero classes are the coordinate class $F_1{+}F_2{+}F_3$, the anchor class A, and their sum. The code that *builds* the lattice (as fiber data of Construction A) reappears after Gaussian reduction as the *message space* of its own lattice — with the fibers redistributed.

4 The quartic companion: G_{31} as the holomorphic μ_4 shadow

All checks of this section are from `quartic_half_probe.py` (22 checks, 0 failures, verdict QUARTIC-HALF-ALIVE; exact Gaussian-integer and Gaussian-rational arithmetic, no floats anywhere), run in the standard doubled-coordinate model of E_8 (roots $2(\pm e_i \pm e_j)$ and $(\pm 1)^8$ with an even number of minus signs; norm 8), which by Remark 3 carries the same bridge. The provable core is formalized in `QuarticHalf.lean`.

4.1 The degree bookkeeping

The exponents of $W(E_8)$ are $\mathrm{Exp}(E_8) = \{1, 7, 11, 13, 17, 19, 23, 29\}$, the totatives of the Coxeter number $h = 30$ [7]. The character χ_{-4} (residue mod 4) splits them: the exponents $\equiv 3 \bmod 4$ are $\{7, 11, 19, 23\}$, and shifted by $+1$,

$$\{7, 11, 19, 23\} + 1 = \{8, 12, 20, 24\} = \mathrm{Deg}(G_{31}),$$

the degree quadruple of the rank-4 complex reflection group G_{31} in the Shephard–Todd classification [3, 6]. Both the split sum and its complement sum equal $7{+}11{+}19{+}23 = 1{+}13{+}17{+}29 = 60$, the number of mirror lines of G_{31}, and $8 \cdot 12 \cdot 20 \cdot 24 = 46080 = |G_{31}|$. Equivalently, the degrees $\mathrm{Deg}\, W(E_8) = (2, 8, 12, 14, 18, 20, 24, 30)$ split mod 4 into the G_{31} half $\{8, 12, 20, 24\}$ ($\equiv 0$) and the complement $\{2, 14, 18, 30\}$ ($\equiv 2$). Conceptually this is Springer's theory of regular elements [5]: J is a ζ_4-regular element of $W(E_8)$ (it acts freely on the roots, so its i-eigenspace contains regular vectors), and the centralizer of J acts on $V^{1,0} = \ker(J - i) \subset \mathbb{C}^8$ ($\dim_{\mathbb{C}} = 4$) as the reflection group with degrees the $W(E_8)$-degrees divisible by 4 — that is, G_{31}. The 60 mirror lines of this action are spanned by the 60 Gaussian root lines of Theorem 2(v), and the probe verifies exactly that the 60 Hermitian reflections in these lines are involutions permuting the 240 roots (check II3.1).

4.2 The restriction theorem

For $d \geq 1$ let $F_d(x) = \sum_\alpha \langle \alpha, x \rangle^d$, summed over the 240 roots, and let $F_d|$ denote the restriction to $V^{1,0}$, computed in the holomorphic weights $\langle \alpha, b_k \rangle$, $b_k = e_{2k} - i e_{2k+1}$. Write $q(x) = \langle x, x \rangle$.

Theorem 6 (quartic invariant restriction, honestly layered)**.** *(a)* (Vanishing half: true but trivial.) *For every $d \not\equiv 0 \bmod 4$, $F_d| = 0$ identically. Proof: the weight law $W(J\alpha) = -i\, W(\alpha)$ holds for all* 240 *roots, so each μ_4-orbit $\{\alpha, J\alpha, -\alpha, -J\alpha\}$ contributes $t^d\big(1 + (-i)^d + (-1)^d + i^d\big) = 0$; the orbit factor vanishes if and only if $d \not\equiv 0 \bmod 4$. This proves the vanishing of the degrees $\{2, 14, 18, 30\}$ — but equally of $6, 10, 22, 26$: the pattern is μ_4-trivial and carries no G_{31} content.*

(b) (Isotropy layer: still no G_{31} content.) *$V^{1,0}$ is q-isotropic, and the symbolic* 8*-variable identities*

$$f_4 = 576\, q^2, \qquad f_6 = 1920\, q^3$$

hold for the (doubled) root power sums — $W(E_8)$ has no basic invariant in degrees $4, 6$ — so $F_4| = F_6| = F_{10}| = 0$ on any *q-isotropic subspace, whether or not the complex structure is root-compatible (the degree-*10 *case because every degree-*10 *invariant is a combination of q^5 and $q f_8$).*

(c) (The load-bearing content.) *The restrictions $F_8|, F_{12}|, F_{20}|, F_{24}|$ are nonzero (exact-point evaluation), invariant under all* 60 *Hermitian reflections and under σ and J, and* algebraically independent *(exact Jacobian $\neq 0$). Since their degrees* $(8, 12, 20, 24)$ *multiply to* $46080 = |G_{31}|$*, Chevalley's theorem [4, 6] makes them a system of* basic invariants *of G_{31}: the restriction of the $W(E_8)$-invariant ring to $V^{1,0}$ generates the full G_{31}-invariant ring.*

(d) (Jacobian mirror factorization.) $\deg \mathrm{Jac}(F_8|, F_{12}|, F_{20}|, F_{24}|) = \sum(d_i - 1) = 60$, *the number of mirror lines; the Jacobian vanishes at exact generic points on each of the* 60 *mirrors and is nonzero off them, consistent with the standard factorization* $\mathrm{Jac} = c \prod_{60}(\textit{mirror forms})$ *[6, Ch. 9] — the full symbolic degree-*60 *expansion was not performed and is recorded as a named residual.*

The certification levels are: (a) is proved symbolically (probe II2.1–II2.2; Lean theorems `weight_pair_J`, `orbit_factor_iff`, `orbit_powersum_vanishes` over $\mathbb{C}$, with the mod-4 degree bookkeeping `deg_we8_mod_four_split`). In (b), $f_4 = 576\,q^2$ is a kernel-checked polynomial identity over *any* commutative ring, assembled from the two explicit root blocks (112 D8-type roots contributing $448 \sum x_i^4 + 384 \sum_{i<j} x_i^2 x_j^2$ and 128 spinor roots contributing $384\,q^2 - 256 \sum x_i^4$; Lean theorems `quartic_d8_block`, `quartic_spinor_block`, `quartic_powersum`), and $F_4| = 0$ follows by the isotropy collapse $1 + (-i)^2 = 0$ (`holomorphic_isotropy`, `quartic_restriction_vanishes`); $f_6 = 1920\,q^3$ and the all-35-coefficients check of $F_4|$ are exact symbolic computations in the probe (II2.3, II2.5). In (c), nonvanishing is rigorous by a nonzero exact value at the Gaussian point $z = (-3+3i,\ -3+2i,\ -3+2i,\ -3)$ for all $d \in \{8, 12, 16, 20, 24, 28\}$ (II2.6 — honestly including the non-degrees $16, 28$: nonvanishing alone is generic mod-4 behaviour; the G_{31} content is invariance + independence); invariance is exact at a $4\mathbb{Z}[i]$-point for all 60 reflections and at 3 exact points for each clock generator (II3.2); independence is the exact Jacobian at 3 exact random points off all mirrors, with 10 small integer points honestly rejected because they landed *on* mirrors, where the Jacobian must vanish (II3.3). In (d) the mirror vanishing is verified at 2 exact generic points on each of the 60 mirrors, 60/60 (II3.5). The generation statement "the 60 reflections generate G_{31}" is cited (Springer theory [5, 6]; repository module `v634`), not re-proved.

4.3 Must-fail controls

A structure claim needs controls that destroy it. Four are run (probe II4, all firing as demanded):

(0) *Honest negative first.* The natural candidate control — flipping the orientation of one pair, $J' = S_7 J S_7$ — does *not* fire: $J' = J \circ S_{67}$ with the even sign flip $S_{67} \in W(D_8) \subset W(E_8)$, so J' preserves all 240 roots. No signed-permutation complex structure can break root compatibility; a genuine control must leave the integral orthogonal group.

(a) $J' = RJR^{\mathsf{T}}$ with R the exact rational $(3, 4, 5)$-rotation crossing two μ_4 pairs: an orthogonal complex structure ($J'^2 = -1$) that preserves only 24/240 roots. The trivial layers persist ($F_2' = F_4' = F_6' = F_{10}' = 0$: isotropy and q-powers, exactly as (b) predicts), but the informative half breaks: $F_d' \neq 0$ for all $d \in \{14, 18, 22, 26, 30\}$. The vanishing pattern of Theorem 6(a) is a property of *root-compatible* complex structures, not of orthogonal complex structures per se.

(b) The root-preserving twin $J'' = S_{57} J S_{57}$ (240/240 roots kept): the full vanishing pattern survives on $\ker(J'' - i)$ — the theorem is about the $W(E_8)$-conjugacy class of J, not one matrix.

(c) A random non-isotropic 4-space: already $F_2 = -56880 - 61440\,i \neq 0$ — even the first layer needs an orthogonal complex structure.

On the bridge side (probe I4): the naive (non-equivariant) placement keeps the $\mathbb{Z}[i]$-module and the census but loses the σ-semantics — σ maps the codeword lift $(0, 0, 0, 1, 1, 1, 1, 0)$ off the lattice, so step (iii) of Theorem 4 is *undefined* there (I4.1–I4.2); 16 of the 30 placements are not even π_J-invariant, and for those no $\mathbb{Z}[i]$-module exists at all (witness root $(-1, -1, -1, 0, -1, 0, 0, 0)$ with J-image off the lattice, I4.3); and on the *other* unimodular Hermitian $\mathbb{Z}[i]$-lattice of rank

4, the standard $\mathbb{Z}[i]^4$, the census trivializes: only 16 minimal vectors, occupying 4 of the 15 nonzero classes with 4 each, 11 classes empty — no 240, no 15×16, no RM$(1,3)$ (I4.4). The bridge needs E_8.

5 Machine verification

5.1 The two probes

probe	checks	verdict	arithmetic
`gaussian_code_bridge_probe.py`	26/26	GAUSSIAN-CODE-BRIDGE-EXACT	exact $\mathbb{Z}$/`Fraction`; SNF via `sympy`
`quartic_half_probe.py`	22/22	QUARTIC-HALF-ALIVE	exact $\mathbb{Z}[i]/\mathbb{Q}(i)$; no floats

Both probes cross-validate their own machinery: the class map is computed twice (Hermite-normal-form labels versus the direct membership test $(1-J)(x-y)/2 \in L$, 300 random cases, I1.5), the census is re-run in a second coordinate model (I5.1), and the Jacobian points are rejection-sampled off the mirrors with the rejections printed (II3.3). Both files are promoted unchanged to the permanent verification suite of the repository [10] as the modules `v689_gaussian_code_bridge` and `v690_quartic_half`.

5.2 The Lean layer

The algebraic cores are formalized in two Lean 4 modules over Mathlib [8, 9], committed to the repository's Lean project (`lake build` green, no `sorry`, no `native_decide`): `GaussianCodeBridge.lean` (44 theorems) and `QuarticHalf.lean` (21 theorems), 65 kernel-checked theorems in total. The finite certificates are proved by the kernel-checked `decide` tactic over explicit integer data — notably the code-equivariance of C^*, the norm-doubling Gram identity $(1+J)^{\mathsf{T}}(1+J) = 2 \cdot \mathbf{1}$, the full Smith certificate with bidirectionally unimodular transforms, the census of the explicit 240 root coordinate vectors (pairwise distinct, norm 4, in the code mod 2; zero class empty; every nonzero label hit exactly 16 times), the μ_4- and σ-transport certificates with the explicit label-block matrix $M_{\bar{\sigma}}$, the family 3-cycle with anchor, and the coordinate-class law $\mathrm{label}(2e_0) = F_1 + F_2 + F_3$. The polynomial identity $f_4 = 576\,q^2$ is proved by `ring` over an arbitrary commutative ring after unfolding the explicit 240-element root list.

Honest scope of the formal layer. The quotient $L/(1+i)L$ is not built as a Mathlib quotient module; the Lean statements concern the explicit Smith residue map and explicit matrices, and the standard SNF argument identifying the quotient is cited, not formalized. Completeness of the 240-element root list (that these are *all* norm-4 vectors) is the probe's exhaustive-enumeration job, re-used, not re-proved, in Lean. The identification of $\bar{\sigma}$ with the RM$(1,3)$ family permutation up to the order-18 gauge, and the entire G_{31} layer of Theorem 6(c)–(d) beyond $f_4 = 576\,q^2$ and the μ_4-orbit mechanism, rest on the probes' exact-point computations plus the cited Chevalley and factorization theorems.

6 Scope and outlook

What is established is finite algebra, fully certified: the pair (E_8, μ_4) carries a canonical four-bit quotient on which the 240 roots equidistribute as 15 messages $\times$ 16 roots (= 4 Gaussian lines) each, with a distinguished coordinate block; the clock σ equips the quotient with the family/anchor semantics of the code that built the lattice; and the holomorphic μ_4 shadow of the $W(E_8)$ invariant ring is exactly the G_{31} invariant ring, with the vanishing half of the degrees typed honestly as the trivial μ_4-orbit factor. We find the combination noteworthy: Construction A uses the $[8,4,4]$ code as *fiber* data, and the Gaussian reduction returns the same

code's four-bit *message* space as an intrinsic invariant of the lattice-with-clock, with the fibers redistributed across the classes (Remark 5). Within the verification program that produced it, this note is typed as structure — an information layer of the E_8 compiler datum — and moves no status marker. No claim is made beyond the stated algebra: no physical readout, no arithmetic consequence, and nothing about any open conjecture is asserted or implied.

References

[1] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Grundlehren der mathematischen Wissenschaften 290, Springer, New York (1999).

[2] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam (1977).

[3] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. 6 (1954), 274–304.

[4] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. 77 (1955), 778–782.

[5] T. A. Springer, *Regular elements of finite reflection groups*, Invent. Math. 25 (1974), 159–198.

[6] G. I. Lehrer and D. E. Taylor, *Unitary Reflection Groups*, Australian Mathematical Society Lecture Series 20, Cambridge University Press, Cambridge (2009).

[7] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4–6*, Hermann, Paris (1968).

[8] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction – CADE 28, Lecture Notes in Computer Science 12699, Springer (2021), 625–635.

[9] The mathlib Community, *The Lean mathematical library*, in: Proc. 9th ACM SIGPLAN Int. Conf. on Certified Programs and Proofs (CPP 2020), ACM (2020), 367–381.

[10] TFPT repository (`https://fixpoint-theory.com`): discovery probes `experiments/tfpt-discovery/gaussian_code_bridge_probe.py` and `quartic_half_probe.py`; suite modules `v689_gaussian_code_bridge`, `v690_quartic_half` (and the predecessor modules `v626`, `v634`, `v638` for the Construction-A code layer, the G_{31} structure and the equivariant placement); Lean 4 proof modules `lean4-carrier-rigidity/TfptCarrier/GaussianCodeBridge.lean` and `QuarticHalf.lean` (2026).