TFPT — Architecture and the E_8 Compiler

The two axioms $\{c_3, g_{\mathrm{car}}\}$, the derivation map, and the Coxeter–cyclotomic $D_5 \times A_3 \to E_8$ construction

June 8, 2026

What this document is about

The architecture layer: how the two axioms $c_3 = \frac{1}{8\pi}$ and $g_{\mathrm{car}} = 5$ build the Coxeter–cyclotomic compiler — the carrier $C^{+}{=}D_5$, the family geometry $\mathbb{P}^1 \setminus \mu_4{=}A_3$, the μ_4 glue $D_5 \oplus A_3 + \mu_4 \Rightarrow E_8$, the EM fixed point α^{-1} (with its ablation), and the whole number alphabet $16, 40, 41, 48, 240, 248$ as carrier traces.

The TFPT document set — what is treated where

Plain language: TFPT is a small discrete compiler. Two inputs — the seam constant $c_3 = \frac{1}{8\pi}$ and the carrier rank $g_{\mathrm{car}} = 5$ — build $D_5 \oplus A_3 + \mu_4 \Rightarrow E_8$ and read off the Standard Model, the constants and the scale grammar. The development is **six short documents**, best read in order:

1. `introduction` — reading guide, compiler closure, paper-by-paper comparison, predictions, the dependency DAG and proof ledger.
2. `tfpt_1_architecture_e8` — the two axioms, the derivation map, the EM fixed point α^{-1}, the $D_5 \oplus A_3 + \mu_4 \Rightarrow E_8$ construction.
3. `tfpt_2_standard_model` — the SM in one φ_0-ladder formula, flavor from parabolic transport, the worked closures, and gravity/QG as the seam response.
4. `tfpt_3_e8_audit_bootstrap` — the seven E_8 slices as an audit raster, the cascade bridge, the Möbius bootstrap.
5. `tfpt_4_frontier` — honest status of η_B, m_p/m_e, Koide, dark matter, full QG.
6. `tfpt_horizon_readouts` — one seam constant as the universal horizon thermal code.

You are reading document #2.

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Part I

Architecture and the derivation map

Abstract

This note reorganises a set of results sitting on top of the existing TFPT 4.5 series (Papers 1–7). It is written for someone familiar with the series and is meant as an *update for comments*, not a finished paper. The thesis is that the bridge layer of TFPT is governed by *three nested grammars*: a **carrier grammar** (from $3+2$ come charges, families, hypercharge, $\Omega_{\mathrm{adm}} = 48$, $b_1 = 41/10$), a **seed grammar** (one seed $u = \varphi_0^{\mathrm{ret}}$ generates the Cabibbo angle, birefringence, the baryon fraction and the reactor angle), and an **action grammar** (the large scale ratios are exponential actions of $\alpha_\star^{-1}$ with rungs $1/g_{\mathrm{car}}$, 1, 2). Two new exact results sharpen the action grammar: the *decuple bridge* $\rho_\Lambda/\bar{M}_{\mathrm{Pl}}^4 = \frac{3}{4\pi^2} e^{2\delta_{\mathrm{ph}}} \left[v_{\mathrm{geo}}/(5\beta_{\mathrm{rad}}^2 \bar{M}_{\mathrm{Pl}}) \right]^{10}$ (the cosmological constant is, in essence, the tenth power of the stripped electroweak scale, with $10 = 2g_{\mathrm{car}}$), and the Hubble corollary $H_0/\bar{M}_{\mathrm{Pl}} = e^{-\alpha_\star^{-1}}/(2\pi\sqrt{\Omega_\Lambda})$. We also give the exact corrected tree form of the Einstein normaliser $\xi = \frac{3}{4}/(1 + 9/128\pi^3)$, the dual-root dictionary that generates all hypercharges and transport cusps from $y_\pm = \{1/2, -1/3\}$, the Λ-anchored metrology closure, the overdetermination of $g_{\mathrm{car}} = 5$, and the E_8 carrier atlas in three clearly separated levels. Section 3 adds the strongest structural result: the carrier code is the $D_5 = \mathfrak{so}_{10}$ half-spinor ($\mathrm{Aut} = W(D_5)$, order 1920; $R_4 =$ Clebsch graph), the family geometry $\mathbb{P}^1 \setminus \mu_4$ is an A_3, and E_8 is the explicit unimodular glue closure of $D_5 \oplus A_3$ — with the discriminant-form norms $q(D_5) = 5/4$ and $q(A_3) = 3/4$ summing to the E_8 root norm 2 and coinciding with the TFPT constants $\delta_2/\delta_{\mathrm{top}}^2$ and ξ_{tree}. Status markers used throughout: **[I]** exact identity, **[L]** Lie/lattice theorem, **[F]** formalised, **[N]** numerical fixed point, **[P]** physical/conditional, **[A]** axiom/open. (All markers are explicit; the legacy `[M]`/`[C]` compatibility aliases have been removed.)

1 How to read this, and where it sits in Papers 1–7

P	Title (short)	What it fixes (used here)
1	Boundary polarization & primitive kernel	$c_3 = 1/(8\pi)$; winding $[u_\Sigma] = 1$
2	Carrier rigidity & SM packet	$g_{\rm car} = 5$, $(b, s) = (3, 2)$, Y, $\dim S^+ = 16$, $N_{\rm fam} = 3$, $\Omega_{\rm adm} = 48$, $b_1 = 41/10$
3	EM fixed point & flavor transport	α, transport pole $\delta_{\rm ph}$, seed u, CKM/PMNS
4	Admissibility, strong CP, QFT closure	selector, $\theta_{\rm eff} = 0$
5	Geometric Hodge closure & metrology	boundary unit λ_Σ, $\bar{M}_{\rm Pl}/\lambda_\Sigma$, $G_N\lambda_\Sigma^2$, ξ
6	Cosmology interfaces	$\Lambda_{\rm IR}$, axion, inflation interface
7	CMB operational closure	Stage-1/2 pipeline (programmatic)

Notation convention. We write $D_5 \oplus A_3$ for the *root-lattice* direct sum (used in the glue/lattice statements: discriminant forms, μ_4 glue, E_8 as an even unimodular lattice), and $D_5 \times A_3$ for the *group/representation* branching (used for subalgebra embeddings and E_8-rep decompositions). The objects are the same atoms; the symbol marks whether the statement is lattice-theoretic ($\oplus$) or branching/representation-theoretic ($\times$).

Nothing here introduces a new primitive assumption; it reorganises existing outputs and proves the cross-links. The whole bridge layer rests on two inputs,

$$\boxed{c_3 = \frac{1}{8\pi} \quad \text{(Paper 1)}, \qquad g_{\rm car} = 5 \quad \text{(Paper 2)}.} \tag{1}$$

Foundational postulates — the two genuine axioms (hardenable, not derivable)

Everything downstream is a theorem *given* two structural inputs that are not reduced further and are honestly flagged as the theory's axioms:

(P1) Boundary kernel.
A single oriented seam Σ carries a primitive, reflection-positive boundary kernel with unit winding $[u_\Sigma] = 1$; its self-linking normalisation fixes $c_3 = 1/(8\pi)$. *Status:* primitive postulate. *Hardenable by:* the Calderón projector / determinant-character chain (Paper 1) and the Lean carrier-rigidity development (`TfptCarrier.*`, boundary-polarization and Calderón interfaces).

(P2) Carrier interface.
This splits into an *axiom* and a *theorem*, which must not be conflated:

- **P2**$_{\rm interface}$ [A] **(axiom):** the seam is read out by a five-slot carrier $g_{\rm car} = 5$ (3 colour + 2 weak) — a postulate (the readout interface).
- **P2**$_{\rm algebra}$ [F] **(theorem):** *given* the carrier, hypercharge ($y_- = -\frac{1}{3}$, $y_+ = \frac{1}{2}$), rigidity and the family count follow from the idempotent/lattice axioms — machine-verified in Lean (`Hypercharge.lean`, `LatticeRigidityGeneral.lean`, `Rigidity.lean`).

So P2 is an *interface axiom* whose *algebraic consequences are a formalised theorem* — "axiom or theorem?" has the precise answer: interface axiom, algebra theorem.

These two are *inputs*: they can be formalised and stress-tested (more Lean, more interface theorems) but not computed away. Every other number in this note — $N_{\rm fam} = 3$, $\Omega_{\rm adm} = 48$, $b_1 = 41/10$, $\alpha_\star^{-1}$, the flavor matrix, the E_8 glue, A_s — is a consequence of (P1),(P2).

Anchor-first refinement. The two inputs are themselves the elementary symmetric polynomials of the single parabolic anchor $a = (1,1,2)$ (the exponents-at-infinity of the splitting type $\mathcal{O}(-2)\oplus\mathcal{O}(-1)^2$): $e_1(a) = 4 = |\mu_4|$, $e_2(a) = 5 = g_{\mathrm{car}}$, $e_3(a) = 2 = |\mathbb{Z}_2|$, with $c_3 = 1/(2e_1(a)\pi) = 1/(8\pi)$. The power sums $p_n(a) = 2+2^n$ then give $|R(E_8)| = p_1p_2p_3 = 240$, $\dim E_8 = 240+(p_4-p_3) = 248$ and the binary ladder $p_{n+1}-p_n = 2^n$. So the input set reduces to $\{a = (1,1,2),\, \pi\}$. [`verification/v23_anchor_generator.py`]

Hardening (P1): the Gauss–Bonnet origin of $c_3 = \frac{1}{8\pi}$

$c_3 = 1/(8\pi)$ is not an arbitrary number: it is the *one-sided Gauss–Bonnet normalisation of the seam.* Paper 5 compactifies the oriented two-dimensional normal slice N_Σ to a sphere S^2, so Gauss–Bonnet gives the seam curvature integral

$$\oint_{S^2} K\,dA = 2\pi\,\chi(S^2) = 4\pi \qquad (\chi(S^2) = 2),$$

and the boundary datum is *one-sided* (the Calderón/double-cover construction uses a single collar), contributing a factor $1/|\mathbb{Z}_2| = \frac{1}{2}$. Hence

$$\boxed{c_3 = \frac{1}{|\mathbb{Z}_2|\,\oint_{S^2} K\,dA} = \frac{1}{2\cdot 4\pi} = \frac{1}{8\pi}}$$

equivalently $c_3 = \frac{1}{2}(4\pi)^{-1}$, the one-sided 2D heat-kernel coefficient. This locates both factors precisely:

- the *discrete* $8 = |\mathbb{Z}_2|\cdot 2\cdot\chi(S^2) = 2\cdot 2\cdot 2$ *coincides with* $h(D_5) = \operatorname{rank} E_8 = \varphi(30)$ — the geometric and the bootstrap readings of the “8” agree;
- the *continuous* π is the irreducible Gauss–Bonnet / 2D-Gaussian primitive ($\int e^{-x^2} = \sqrt{\pi}$), the single primitive the self-consistency loop also isolates.

Status: this *hardens* (P1) — it gives c_3 a named geometric origin and reconciles the “8” with E_8 — but does *not* eliminate π; π remains the one continuous primitive. **[P]**

The one-paragraph version

There is a boundary (the seam). It yields $c_3 = 1/(8\pi)$ and a five-slot carrier $g_{\mathrm{car}} = 5$, split 3 colour $+\,2$ weak. Two base charges $y_- = -1/3$ and $y_+ = 1/2$ generate the whole fermion family; the 16 even-occupation states are one generation, three families give $3\times 16 = 48$. One seed $u = \varphi_0^{\mathrm{ret}}$ drives four low-energy observables. Finally $\alpha_\star^{-1} \approx 137$ is the exponential engine: the electroweak scale takes one fifth of it ($e^{-\alpha_\star^{-1}/5}$), the cosmological constant takes it twice ($e^{-2\alpha_\star^{-1}}$), and the Hubble scale is the square root of Λ. A small grammar generates many sectors.

2 Headline: the Pascal compiler on five carrier slots

The strongest new statement is that three of the theory’s separate structures are the *same* truncated row of Pascal’s triangle on the five-slot carrier.

Family = action ladder = transition count (one Pascal row)

The even exterior algebra of E ($\dim E = g_{\rm car} = 5$), the action ladder, and the E_8 root count are one Pascal row:

$$S^+ = \Lambda^0 E \oplus \Lambda^2 E \oplus \Lambda^4 E, \qquad (\dim \Lambda^0, \dim \Lambda^2, \dim \Lambda^4) = (1, 10, 5), \qquad \dim S^+ = 1{+}10{+}5 = 16,$$

$$\mathcal{A}_{\rm EW} : \mathcal{A}_H : \mathcal{A}_\Lambda = \binom{5}{0} : \binom{5}{1} : \binom{5}{2} = 1 : 5 : 10 = (\dim \Lambda^0 E,\ \dim \Lambda^4 E,\ \dim \Lambda^2 E),$$

$$|R(E_8)| = \dim S^+(\dim S^+ - 1) = 16 \cdot 15 = 240.$$

Family (exterior degree), cosmology (action charge) and E_8 (root transitions) read the same $\{1, 5, 10\}$. [I]

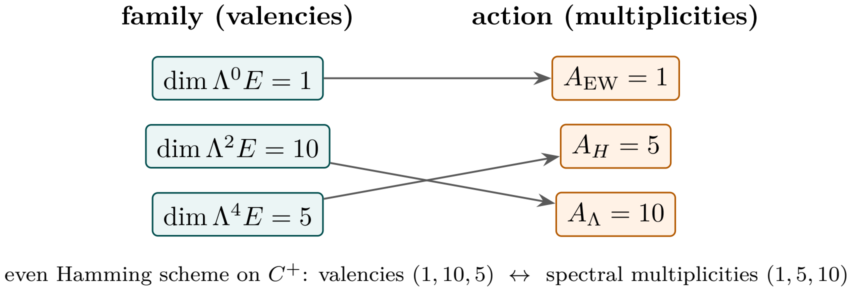


Figure 1: Exterior–Action duality. The Standard-Model family is the valency side of the even Hamming scheme on the five-slot code; the cosmological action ladder is its spectral (multiplicity) side. The Λ (pair) sector $\Lambda^2 E$ has dimension $\binom{5}{2} = 10$, the origin of the decuple exponent.

Natural sector assignment:

$$\boxed{EW \leftrightarrow \Lambda^0 E \text{ (singlet)}, \quad H \leftrightarrow \Lambda^4 E \cong E^\vee \ (\dim 5), \quad \Lambda \leftrightarrow \Lambda^2 E \ (\dim 10).}$$

The cosmological constant lives on the *pair* sector $\Lambda^2 E$, dim $= \binom{5}{2} = 10$; this is why the decuple bridge has exponent 10 — Λ is a product over all $\binom{5}{2}$ carrier pairs (K_5 edges), not over the full 16-state family.

2.1 $g_{\rm car} = 5$ as the Pascal closure

A family is exactly the complete Pascal truncation to degree two:

$$\boxed{\dim S^+ = 2^{g_{\rm car}-1} = \binom{g_{\rm car}}{0} + \binom{g_{\rm car}}{1} + \binom{g_{\rm car}}{2} \iff g_{\rm car} = 5} \qquad (16 = 1 + 5 + 10), \quad \text{[I]}, \quad (2)$$

the unique positive-integer solution of $f(g) := 2^{g-1} - 1 - \frac{g(g+1)}{2} = 0$. *Proof of uniqueness.* Direct evaluation gives $f(1){=}{-1}$, $f(2){=}{-2}$, $f(3){=}{-3}$, $f(4){=}{-3}$, $f(5){=}0$, $f(6){=}{+10}$; so $g{=}5$ is the only root in $\{1, \ldots, 6\}$. For $g \geq 6$ the forward difference is $f(g{+}1){-}f(g) = 2^{g-1}{-}(g{+}1) \geq 2^5{-}7 = 25 > 0$ and increasing, so f is strictly increasing on $[6, \infty)$ from $f(6){=}10 > 0$; hence $f(g) > 0$ for all $g \geq 6$ and there is *no* further root. Thus $g_{\rm car} = 5$ is the unique solution (no finite-range caveat). [`verification/v2_carrier_pascal.py`] This is sharper than $2g_{\rm car} = \binom{g_{\rm car}}{2}$: the family code (even states, 2^{g-1}) equals the action code (sum of the first three Pascal entries) only at $g_{\rm car} = 5$.

Carrier uniqueness theorem: $g_{\text{car}} = 5$ and the 3+2 split are forced [I]

The carrier rank and its colour/weak split are not chosen — they are the unique solutions of two integer conditions tied to the E_8 closure and the SM gauge algebra:

$$\boxed{g_{\text{car}} = 5 \text{ is the unique } g \text{ with } N_{\text{fam}} = \tfrac{2^{g-1}-1}{g} \in \mathbb{Z}^+ \text{ and } \operatorname{rank} E_8 = g + N_{\text{fam}} = 8} \tag{3}$$

$$\boxed{(b, s) = (3, 2) \text{ is the unique split of } \ b + s = g_{\text{car}} = 5, \ \ b^2 + s^2 = 13 = |R(A_3)|{+}1} \tag{4}$$

The integer-family carriers are $g \in \{3, 5, 7, \dots\}$ (rank $4, 8, 16$), so rank $E_8 = 8$ picks $g_{\text{car}} = 5$ uniquely; and $b^2 + s^2 = 13$ with $b{+}s = 5$ forces $(3, 2)$, reproducing $\dim \mathfrak{g}_{\text{SM}} = (b^2{-}1) + (s^2{-}1) + 1 = 12 = |R(A_3)|$. The induced hypercharge is anomaly-free ($\operatorname{Tr} Y = \operatorname{Tr} Y^3 = 0$ over the 16 states). This hardens P2: the *rank* and the *split* are forced; only the seam→carrier interface itself remains a declared input. [`verification/v14_carrier_uniqueness.py`]

The Pascal grammar: one $\Lambda^\bullet$ mechanism in four places (anti-numerology) [I]

The recurring small integers are *not* coincidences — they are the *same* exterior-algebra (Pascal) mechanism read in four spaces, which is why the binomials of rows 4 and 5 keep reappearing:

space	**Pascal reading**
carrier (one generation)	$\dim S^+ = \Lambda^{\text{even}}(5) = \binom{5}{0}{+}\binom{5}{2}{+}\binom{5}{4} = 1{+}10{+}5 = 16$
scale ladder (cosmology)	$A_{\text{EW}} : \mathcal{A}_H : \mathcal{A}_\Lambda = \binom{5}{0} : \binom{5}{1} : \binom{5}{2} = 1 : 5 : 10$
flavor u/d readout	$\lVert \operatorname{Pl}(K) \rVert_1 = \binom{4}{0}{+}\binom{4}{1}{+}\binom{4}{2} = 1{+}4{+}6 = 11 = 16{-}g_{\text{car}}$
mass volume	$\det M_{\text{SM}} \sim (\varphi_0^{\text{ret}})^{6+9+10} = (\varphi_0^{\text{ret}})^{25} = (\varphi_0^{\text{ret}})^{g_{\text{car}}^2}$

So the carrier is the even row 5, the scale ladder is the lower row 5, the u/d leg is the cumulative row 4 up to $\deg u^c = 2$, and the mass volume closes on g_{car}^2. *One* grammar (Λ^k of the 5-slot carrier and the $4 = |\mu_4|$ family fundamental), not four lucky hits. [`verification/v44_carrier_exterior.py`] [`verification/v45_family_exterior.py`] [`verification/v46_grand_mass_volume.py`]

2.2 One master moment generates 40, 41, 240, δ_2

With $X = 6Y$, the quadratic hypercharge trace on one family is $\operatorname{Tr}_{S^+} X^2 = 120 = 5!$, and it generates the whole integer chain:

$$\begin{gathered} \sum_{f,j} L_{f,j} = \frac{\operatorname{Tr}_{S^+} X^2}{3} = 40, \qquad 10 b_1 = \frac{\operatorname{Tr}_{S^+} X^2}{3} + 1 = 41, \\ |R(E_8)| = 2 \operatorname{Tr}_{S^+} X^2 = 240, \qquad \delta_2 = 4!\ \operatorname{Tr}_{S^+} X^2\, c_3^8. \end{gathered} \tag{5}$$

The flavor budget, the $U(1)$ budget, the E_8 count and the second seam defect all hang on the second hypercharge moment 5!. [I]

2.3 Compact E_8 compression and the role of ν^c

$$\boxed{\begin{aligned} &|R(E_8)| = \dim S^+ \cdot g_{\text{car}} \cdot N_{\text{fam}} = 16 \cdot 5 \cdot 3 = 240, \\ &\dim E_8 = \dim S^+ (\mathcal{A}_H + \mathcal{A}_\Lambda) + (g_{\text{car}} + N_{\text{fam}}) = 240 + 8 = 248 \end{aligned}} \tag{6}$$

with $N_{\text{fam}} = (2^{g_{\text{car}}-1} - 1)/g_{\text{car}} = 15/5 = 3$ and rank $E_8 = g_{\text{car}} + N_{\text{fam}} = 8$. [I]. The right-handed neutrino ν^c is the Pascal singleton $\Lambda^0 E = (1,1)_0$ that closes $16 = 1 + 5 + 10$; removing it breaks the truncation and shifts α^{-1} by 2.16×10^{-3}.

3 The $D_5 \times A_3$ compiler: E_8 as a closure, not an input

This section upgrades the E_8 atlas (Section 10) from a counting coincidence to an explicit lattice construction. The whole content is consolidated and verified in the standalone companion note *"The Even Carrier Code and the $\mathbb{Z}_{30}$ Coxeter Compiler."* The thesis: the carrier code is the D_5 half-spinor, the family geometry is an A_3, and E_8 is the standard unimodular glue closure of $D_5 \oplus A_3$ — with the four punctures μ_4 as the glue code.

The compiler in one line

$$\boxed{\text{five carrier slots} \Rightarrow D_5, \quad \text{four family corners } \mu_4 \Rightarrow A_3, \quad (D_5 \oplus A_3) + \mu_4\text{-glue} \Rightarrow E_8.}$$

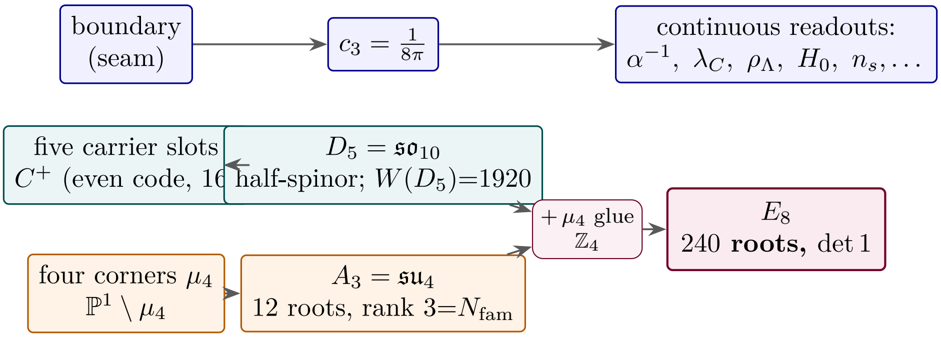


Figure 2: The explanation hierarchy. The boundary fixes the seam seed c_3; the five-slot even carrier code is the D_5 half-spinor; the four punctures μ_4 give A_3; their common $\mathbb{Z}_4$ discriminant (μ_4) glues $D_5 \oplus A_3$ into the unimodular E_8. E_8 is the *closure*, not the input; c_3 dresses the continuous observables.

3.1 Carrier $= D_5 = \mathfrak{so}_{10}$ (proved)

The 16 even-occupation states $C^+ = \{x \in \mathbb{F}_2^{g_{\text{car}}} : |x| \text{ even}\}$ are exactly the weights of the D_5 half-spinor $\frac{1}{2}(\pm 1, \ldots, \pm 1)$ (even number of minus signs). The flip-weight-4 relation is the **Clebsch graph** SRG$(16, 5, 0, 2)$ and the flip-weight-2 relation is its complement SRG$(16, 10, 6, 6)$; the *affine* automorphism group of the scheme (translations $\rtimes$ coordinate permutations) is the D_5 Weyl group,

$$\boxed{\text{Aut}_{\text{aff}}(C^+) = C^+ \rtimes S_5, \quad |\text{Aut}_{\text{aff}}(C^+)| = 2^{g_{\text{car}}-1} g_{\text{car}}! = 1920 = |W(D_5)| = r\,|R(E_8)|} \qquad [\text{I}], \tag{7}$$

(the linear code automorphism group alone is only S_5; the order-1920 group is the affine/scheme one) and the Clebsch edge count is the flavor budget $|E(R_4)| = \frac{16 \cdot 5}{2} = 40 = |R(D_5)| = \sum_{f,j} L_{f,j}$.

3.2 Family $= A_3 = \mathfrak{su}_4$ from $\mathbb{P}^1 \setminus \mu_4$ (proved, with the metric settled)

Paper 2 fixes $X_f^\circ \cong \mathbb{P}^1 \setminus \mu_4$, $H_1 = \mathbb{Z}^3$, rigid D_4 generated by $z \mapsto iz$, $z \mapsto 1/z$. The twelve puncture-difference cycles $\gamma_i - \gamma_j$ carry the residue pairing $\text{Res}(\omega_i, \gamma_j) = \delta_{ij}$ and form the A_3 root system:

$$\boxed{\#\{\gamma_i - \gamma_j\} = 12 = |R(A_3)| = \dim \mathfrak{g}_{\text{SM}}, \quad \text{rank} = 3 = N_{\text{fam}}, \quad \text{Gram} = \text{Cartan}(A_3),\ \det = 4} \qquad [\text{I}]. \tag{8}$$

The corner rotation $z \mapsto iz$ acts on $H^1 = \mathbb{C}^3$ as the A_3 Coxeter element (order $4 = h(A_3) = |\mu_4|$, spectrum $\{i, -1, -i\}$, $\chi = \Phi_4\Phi_2$), an isometry of the Cartan form — the exact analogue of the E_8 Coxeter element reading Φ_{30}.

The metric, settled (former [P])

The transcendental geometric (Green) metric $G_{ij} = -\log|p_i - p_j|$ on μ_4 is only D_4-symmetric, with a regularisation-independent anisotropy $-\log 2$ on the root space, so it is *not* proportional to the A_3 Cartan form. But it does not need to be: the E_8-relevant A_3 is the canonical *residue* pairing (integral, S_4-symmetric, $=\text{Cartan}(A_3)$), and the geometric metric is a D_4-equivariant deformation preserving the root system, Weyl group and Coxeter element. So the lattice structure is exact [I]; the analytic metric is a characterised deformation, not an open gap.

3.3 The μ_4 glue and the $5/4 + 3/4 = 2$ discriminant theorem (proved)

The two factors have equal discriminant groups and E_8 is unimodular, so the glue index is exactly $|\mu_4|$:

$$\text{disc}(D_5) = \text{disc}(A_3) = \mathbb{Z}_4, \qquad [E_8 : D_5 \oplus A_3] = \sqrt{\frac{\det D_5 \det A_3}{\det E_8}} = \sqrt{\frac{4\cdot 4}{1}} = 4 = |\mu_4|. \tag{9}$$

The discriminant-form norms sum to the E_8 root norm, and — strikingly — they are pre-existing TFPT constants:

$$\boxed{\begin{aligned} &q(D_5) = \tfrac{5}{4} = \frac{\delta_2}{\delta_{\text{top}}^2}, \qquad q(A_3) = \tfrac{3}{4} = \xi_{\text{tree}}, \\ &q(D_5) + q(A_3) = 2 = |E_8 \text{ root}|^2 \end{aligned}} \qquad \text{[I]} \ \left[\texttt{verification/v1\_e8\_glue.py}\right] \tag{10}$$

This is literally the E_8 spinor-root split $\frac{1}{2}(\pm)^8 = \frac{1}{2}(\pm)^5 \oplus \frac{1}{2}(\pm)^3$, $5 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = 2$.

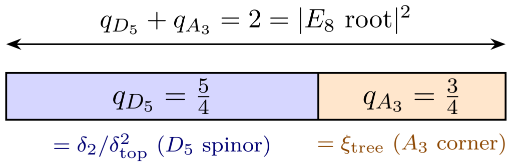


Figure 3: The discriminant-norm glue. The E_8 spinor root $\frac{1}{2}(\pm)^8$ (norm 2) splits across $D_5 \times A_3$ as $5 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4}$; the two halves are the pre-existing TFPT constants $\delta_2/\delta_{\text{top}}^2 = \frac{5}{4}$ and $\xi_{\text{tree}} = \frac{3}{4}$.

The standard branching is realised explicitly:

$$\mathfrak{e}_8 = (45,1)\oplus(1,15)\oplus(10,6)\oplus(16,4)\oplus(\overline{16},\overline{4}), \qquad 248 = 45{+}15{+}60{+}64{+}64, \qquad 240 = 40{+}12{+}60{+}64{+}64, \tag{11}$$

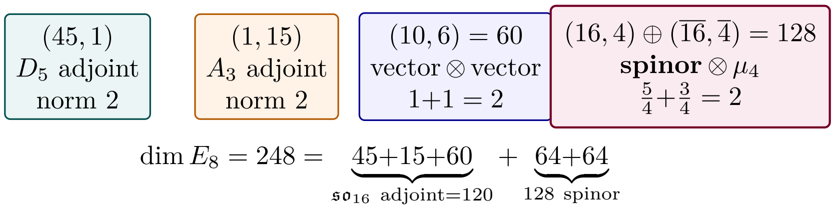


Figure 4: The $E_8 \supset D_5 \times A_3$ branching as norm-2 blocks. The adjoint half is the hypercharge moment $\text{Tr}_{S^+} X^2 = 120 = \dim \mathfrak{so}_{16}$; the spinor half is two sheets over the four corners μ_4.

and assembling $D_5 \oplus D_3 \subset D_8$ with the even spinor coset produces all 240 roots (norm 2, Cartan det $= 1$, every root an integer combination of one simple-root basis): *the* 240*-metrisation is no longer blind — E_8 is built.* [I]

3.4 Glue-norm arithmetic, and three readings of 41, 60, 120

The two glue norms generate the carrier integers through sum, difference and product. With $\dim S^+ = 16$:

$$\boxed{\begin{array}{l} 16(q_D{+}q_A) = 32 = 2^{g_{\rm car}} = h(D_5)h(A_3), \qquad 16(q_D{-}q_A) = 8 = r(E_8) = h(D_5), \\ 16\, q_D q_A = 15 = \dim S^+{-}1 \end{array}} \quad [\mathrm{I}], \qquad (12)$$

so the glue already contains the full carrier exhaustion $2^{g_{\rm car}}$, the E_8 rank, and the non-trivial transition count $\mathcal{A}_H{+}\mathcal{A}_\Lambda = 15$; and $|\mu_4| = (q_D - 1)^{-1} = (1 - q_A)^{-1} = 4$ is the inverse displacement of the two norms from the unit norm.

Spine Quotient Lemma: the recurring factors are adjacent quotients of $(2, 3, 4, 5)$ [I] ([`verification/v74_compiler_micro_lemmas.py`])

The integers of the spine are $(|\mathbb{Z}_2|, N_{\rm fam}, |\mu_4|, g_{\rm car}) = (2, 3, 4, 5)$. Every "mysterious" $O(1)$ factor in TFPT is an *adjacent quotient* of this chain, not a separate special number:

$$\frac{|\mathbb{Z}_2|}{N_{\rm fam}} = \frac{2}{3}, \qquad \frac{N_{\rm fam}}{|\mu_4|} = \frac{3}{4} = q(A_3), \qquad \frac{|\mu_4|}{g_{\rm car}} = \frac{4}{5}, \qquad \frac{g_{\rm car}}{|\mu_4|} = \frac{5}{4} = q(D_5), \qquad \frac{|\mu_4|}{N_{\rm fam}} = \frac{4}{3}.$$

These are exactly the factors that recur across the theory: $\lambda_2 = (\frac{2}{3})^6$ (the transport gap $=$ Page recovery rate, and the Koide target $Q_\star = \frac{2}{3}$); $\varepsilon = \frac{3}{4}\varphi_0 = q(A_3)\varphi_0$ (the solar misalignment); $\varphi_0 = \frac{4}{3}c_3 + 48c_3^4$ with leading $\frac{4}{3}c_3 = \frac{1}{6\pi}$; and $q(D_5){+}q(A_3) = \frac{5}{4}{+}\frac{3}{4} = 2$ (the E_8 root norm). So $(\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{4}, \frac{4}{3})$ are five projections of one discrete chain — not scattered constants.

The carrier adjoint $\dim D_5 = |R(D_5)| + g_{\rm car} = 45$ then gives a second reading of the $U(1)$ integer,

$$\boxed{10b_1 = \dim D_5 - |\mu_4| = 45 - 4 = 41 = \underbrace{40}_{\sum L} + \underbrace{(g_{\rm car} - |\mu_4|)}_{=N_\Phi=1}} \qquad [\mathrm{I}], \qquad (13)$$

i.e. the single Higgs doublet is the carrier rank minus the glue corners, $5 - 4 = 1$. The start defect 60 and the hypercharge moment 120 are "bilingual":

$$\begin{gathered} D_{\rm start} = \dim D_5 + \dim A_3 = 45 + 15 = 60 = \mathcal{A}_\Lambda \cdot |R^+(A_3)| = 10 \cdot 6, \\ \mathrm{Tr}_{S^+} X^2 = 120 = \underbrace{60}_{\text{adjoint}} + \underbrace{60}_{\text{mixed (10,6)}}, \end{gathered} \qquad (14)$$

and $\dim E_8 = 248 = \mathrm{Tr}_{S^+} X^2 + 2\dim S^+|\mu_4| = 120 + 2 \cdot 16 \cdot 4$. Finally the hypercharge norm and the EW exponent are A_3-rational,

$$\boxed{\gamma_{\rm car} = \frac{\mathcal{A}_\Lambda}{|R(A_3)|} = \frac{10}{12} = \frac{5}{6}, \qquad I_1^{\rm EW} = \frac{|E(K_5)|}{|R(A_3)|} + \frac{1}{|\mu_4||R(A_3)|} = \frac{10}{12} + \frac{1}{48} = \frac{41}{48}} \qquad [\mathrm{I}]. \qquad (15)$$

Transport reduction. The flavor matrix L has its mod-6 content fixed by a single combinatorial datum: $\{0, 1, 3\}$ is the *unique* (up to D_6) 3-subset of $C_6 = |R^+(A_3)|$ with all distinct cyclic distances $\{1, 2, 3\}$, and the three sectors are its D_6 orbit ($u{\equiv}\{0, 1, 3\}$, $d{\equiv}2{-}\{0, 1, 3\}$, $e{\equiv}2{+}\{0, 1, 3\}$). The total is fixed by the carrier, $\sum L = 40 = |R(D_5)|$, with winding unit $6 = |R^+(A_3)|$. Given the residue *set* of a sector, the full ordered packet is reproduced by two universal rules — the lightest generation carries the single $+6$ winding, and the entries are sorted in decreasing length (monotone hierarchy). The *only* remaining datum is then, per sector, which residue is the first-generation anchor that receives the winding:

$$\boxed{\text{anchors } (a_u, a_d, a_e) = (1, 1, 2) \;\Rightarrow\; L_u = (7, 3, 0),\; L_d = (7, 5, 2),\; L_e = (8, 5, 3)} \qquad [\mathrm{I}] \text{ (given anchors).} \qquad (16)$$

So the open step of a full “Clebsch walk” theorem is sharply isolated: it is exactly these three per-sector anchors. They are *not* free: the three transport cusps are exactly the D_5 dual-root combinations, and each sector is tied to its cusp by the right-handed hypercharge,

$$\boxed{\{1, \tfrac{2}{3}, \tfrac{1}{3}\} = \{2y_+,\ -2y_-,\ 2(y_+ + y_-)\}, \qquad \text{lepton} \leftrightarrow 2y_+,\ \text{up} \leftrightarrow -2y_-,\ \text{down} \leftrightarrow 2(y_+ + y_-)} \quad \text{[I]}. \tag{17}$$

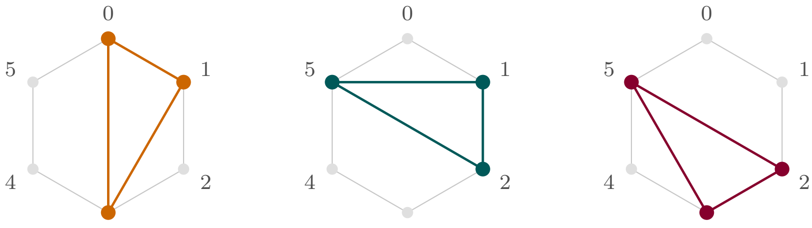
Paper 3 (Thm. “canonical transport kernel”) derives the residue packets — and hence the anchors — as the *word-lengths of the family-connection holonomy* $\mathrm{Hol}_{\mathcal{G}_{D_4}}(\nabla^\star_F)$ along the D_4-invariant geodesic spine of $\mathbb{P}^1 \setminus \mu_4$ with these cusps as boundary data. The anchors are therefore a genuine $D_5 \times A_3$ readout (cusps from the D_5 carrier, spine holonomy from the A_3 four-punctured sphere), not an extra parameter. A compact candidate closed form is the nearest-integer cusp map

$$\boxed{a_f = \mathrm{round}(2\,\mathrm{cusp}_f) : \quad (a_u, a_d, a_e) = \mathrm{round}(\tfrac{4}{3}, \tfrac{2}{3}, 2) = (1, 1, 2)} \qquad \text{[P]}, \tag{18}$$

which reproduces the matrix exactly. The remaining open step is now precise and purely geometric: an *independent* closed-form evaluation of the D_4-spine holonomy word-lengths from the lattice data alone (the same $\mathbb{P}^1 \setminus \mu_4$ whose residue pairing is Cartan(A_3) and whose corner rotation is the A_3 Coxeter element), rather than from the Paper-3 family-connection theorem.

Riemann–Hilbert check, and what it does (and does not) fix

Setting up the rigid $\mathbb{Z}_4$-symmetric rank-3 $SU(3)$ local system explicitly — four puncture monodromies $M_k = R^k C R^{-k}$ with the order-3 cusp class $C \sim \mathrm{diag}(1, \omega, \omega^2)$ ($\omega = e^{2\pi i/3}$, the three cusps as eigenvalue-exponents) and the rotation R ($R^4 = \mathbb{1}$) — the closure $M_0 M_1 M_2 M_3 = \mathbb{1}$ has a solution to machine precision, and the unique R has eigenvalues $\{i, -1, -i\}$: the A_3 Coxeter element. This *independently* confirms the rigidity of Paper 2’s family connection and the corner-rotation = Coxeter identity. However, the flat monodromy *alone* does not fix the integer word-lengths $\{0, 1, 3\}$: those are *parabolic degrees* (the cusp exponents as parabolic weights, together with the $\mathbb{Z}_2$ sheet parity), i.e. Hodge/metric data on the spine, not topological monodromy. So the residual [A] is now identified sharply as a *parabolic-weight computation* on $\mathbb{P}^1 \setminus \mu_4$ — the cusp exponents $\{0, \tfrac{1}{3}, \tfrac{2}{3}\}$ promoted to parabolic degrees — not a free choice and not a pure local-system question. [A]



up ≡ $\{0, 1, 3\}$ down ≡ $2-\{0, 1, 3\}$ lepton ≡ $2+\{0, 1, 3\}$

Figure 5: Flavor transport on the hexagon $C_6 = Z_6 = |R^+(A_3)|$. The unique distinct-distance triple $\{0, 1, 3\}$ (cyclic distances $\{1, 2, 3\}$) and its D_6 orbit give the three sectors’ residues mod 6; the total $\sum L = 40 = |R(D_5)|$ and the first-generation $+6$ winding come from the carrier.

3.5 What the compiler outputs for the Standard Model and our reality

Every entry below is an *output* of (c_3, g_{car}) through the $D_5 \times A_3$ compiler; none is an independent input. “Reads” names the lattice object the observable comes from.

SM / cosmology observable	Compiler output	Reads	St.
Number of generations	$N_{\mathrm{fam}} = 3 = \operatorname{rank} A_3 = \dim H^1(\mathbb{P}^1 \setminus \mu_4)$	A_3	[I]
Gauge algebra dimension	$\dim \mathfrak{g}_{\mathrm{SM}} = 12 = \lvert R(A_3)\rvert = \gcd(6r, 2h)$	A_3	[I]
One generation (16, incl. ν^c)	$S^+ = D_5$ half-spinor $= \Lambda^{0,2,4}E$	D_5	[I]
Hypercharge spectrum	all Y from $y_\pm = \{\frac{1}{2}, -\frac{1}{3}\}$; $\operatorname{Tr} X^2 = 120$	D_5 root data	[I]
Abelian index b_1	$10b_1 = 1 + \lvert\mu_4\rvert\lvert E(K_5)\rvert = 1 + 4 \cdot 10 = 41$	$\mu_4 \times K_5$	[I]
Fermion family occupancy	$\Omega_{\mathrm{adm}} = N_{\mathrm{fam}} \dim S^+ = (N_{\mathrm{fam}} + N_\Phi)\lvert R(A_3)\rvert = 48$	A_3	[I]
Fine structure $\alpha_\star^{-1} = 137.0359992168$	root of $F_{U(1)}(\alpha) = 0$ (explicit, unique)	EM closure	[I]
Cabibbo angle	$\lambda_C = \sqrt{u(1-u)}$, $u = \frac{4}{3}c_3 + 48c_3^4$	seed	[I]
CKM CP phase	$\delta_{\mathrm{CKM}} = \frac{2\pi}{\lvert R^+(A_3)\rvert} + N_{\mathrm{fam}} u(1-u) = \frac{\pi}{3} + 3\lambda_C^2$	A_3^+, seed	[I]/[P]
PMNS angles/phase	$\sin^2\theta_{12} = \frac{1}{3} - \frac{u}{2}$, $\sin^2\theta_{13} = e^{-5/6}u$, $\delta^\nu = \frac{4\pi}{3}$	A_3^+, seed	[I]/[P]
Charged-lepton hierarchy	$\det M_\ell \propto u^{\,h/N_{\mathrm{fam}}} = u^{10}$; $(5,3,2)$	decuple	[I]/[P]
Mass-hierarchy transport	$\sum L = 40 = \lvert R(D_5)\rvert$; entries on $C_6 = \lvert R^+(A_3)\rvert$	$D_5 \times A_3$	[I]
Strong CP	$\theta_{\mathrm{eff}} = 0$ (determinant erasure, Paper 4)	—	[I]
Weak CP	holonomy: $\delta_{\mathrm{CKM}} - \frac{\pi}{3} = N_{\mathrm{fam}}\, u(1-u)$	A_3^+	[P]
Gravity tree normaliser	$\xi_{\mathrm{tree}} = \frac{\dim \mathfrak{g}_{\mathrm{SM}}}{\dim S^+} = \frac{3}{4} = q(A_3)$	A_3 glue	[I]
Cosmological constant	$\rho_\Lambda/\bar{M}_{\mathrm{Pl}}^4 = \frac{3}{4\pi^2} e^{2\delta_{\mathrm{ph}}}[v_{\mathrm{geo}}/(5\beta_{\mathrm{rad}}^2 \bar{M}_{\mathrm{Pl}})]^{10}$	decuple	[I]
Hubble anchor	$H_0\sqrt{\Omega_\Lambda} = \bar{M}_{\mathrm{Pl}} e^{-\alpha_\star^{-1}}/(2\pi)$	action	[I]
Baryon density	$\omega_b = \lambda_C/\mathcal{A}_\Lambda = \lambda_C/10$	decuple	[I]/[P]
Inflation (scalaron)	$M_{\mathrm{scal}}^2/\bar{M}_{\mathrm{Pl}}^2 = c_3^{\Omega_{\mathrm{adm}} - 10b_1} = c_3^7$	deficit $r-1$	[I]
Inflation amplitude	$A_s = N_\star^2 c_3^7/24\pi^2 \approx 2.2 \times 10^{-9}$	deficit	[P]/[A]

*Marker key: **[I]** exact identity · **[L]** Lie/lattice theorem · **[F]** formalised · **[N]** numerical fixed point · **[P]** physical/conditional · **[A]** axiom / open.*

Reading. The discrete sector — generations, gauge dimension, family content, hypercharge, b_1, occupancy — is now *group-theoretic output* of $D_5 \times A_3$. The continuous sector — α^{-1}, λ_C, the mixings, ρ_Λ, H_0, inflation — is the seam seed c_3 and the action exponentials read on the *same* carrier graphs (K_5 pairs, the C_6 hexagon $= \lvert R^+(A_3)\rvert$, the decuple $\mathcal{A}_\Lambda = \binom{g_{\mathrm{car}}}{2} = 10$). What used to be a list of separate hits is, structurally, one machine: D_5 carries the family and the flavor budget, A_3 carries the generations and the gauge dimension, μ_4 glues them (and reappears as the 41 chain, the discriminant norms, and the CP phase step $2\pi/6$), and c_3 dresses everything.

4 Grammar I — the carrier grammar $(3+2)$

From $g_{\mathrm{car}} = 5$ and the split $(b, s) = (3, 2)$ the Paper-2 data follow:

$$\begin{aligned} \gamma_{\mathrm{car}} &= \frac{g_{\mathrm{car}}}{g_{\mathrm{car}}+1} = \frac{5}{6}, \quad N_{\mathrm{fam}} = \frac{g_{\mathrm{car}}+1}{2} = 3, \quad \dim S^+ = 2^{g_{\mathrm{car}}-1} = 16, \\ \Omega_{\mathrm{adm}} &= N_{\mathrm{fam}}\, 2^{g_{\mathrm{car}}-1} = 48, \qquad b_1 = \frac{g_{\mathrm{car}}\, 2^{g_{\mathrm{car}}-2} + 1}{10} = \frac{41}{10}. \end{aligned} \tag{19}$$

4.1 Dual-root dictionary (everything from two charges)

Two carrier roots generate the family and the transport cusps [I]

With $y_+ = \frac{1}{2}$ and $y_- = -\frac{1}{3}$ (the determinant-normalised carrier eigenvalues), the full one-family hypercharge spectrum and the flavor-transport cusps are sums of the two roots:

$$\begin{aligned} Y(Q_L) &= y_+ + y_- = \tfrac{1}{6}, & Y(u^c) &= 2y_- = -\tfrac{2}{3}, & Y(d^c) &= -y_- = \tfrac{1}{3}, \\ Y(L_L) &= -y_+ = -\tfrac{1}{2}, & Y(e^c) &= 2y_+ = 1, & Y(\nu^c) &= 0, \end{aligned}$$

$$\{1, \tfrac{2}{3}, \tfrac{1}{3}\} = \{\, 2y_+,\ -2y_-,\ 2(y_+ + y_-)\,\}, \qquad \gamma_{\rm car} = \mathrm{Tr}_E\, Y^2 = 3y_-^2 + 2y_+^2 = \tfrac{5}{6},$$

$$\varphi_{\rm base} = \frac{1-\gamma_{\rm car}}{\pi} = \frac{1}{6\pi}.$$

All residuals are exactly zero. The Standard-Model charge table is not assumed; it is built by addition from $\{1/2, -1/3\}$.

The $X = 6Y$ quadratic and the hypercharge Pascal-sign duality [I]

Clearing denominators with $X = 6Y$ turns the two base charges into the integer roots $x_+ = 3$, $x_- = -2$ of

$$\boxed{x^2 - x - 6 = 0}: \qquad x_+ + x_- = 1 = N_\Phi, \quad x_+ - x_- = 5 = g_{\rm car},$$

$$-x_+ x_- = 6 = |R^+(A_3)|, \qquad \sqrt{\Delta} = 5,$$

so the Higgs singleton, the carrier rank and the A_3 hexagon all fall out of one quadratic. Counting the signs of X over a generation ($X = 1^{\times 6}, 2^{\times 3}, 6^{\times 1}, (-4)^{\times 3}, (-3)^{\times 2}, 0^{\times 1}$) gives the *Pascal triple* again:

$$\boxed{(\#X{=}0,\ \#X{<}0,\ \#X{>}0) = (1, 5, 10) = \tbinom{5}{0}, \tbinom{5}{1}, \tbinom{5}{2}},$$

$$10{-}5{=}g_{\rm car}, \quad 5{-}1{=}|\mu_4|, \quad 10{-}1{=}\,\mathrm{tr}\, R,$$

with $\nu^c \leftrightarrow \binom{5}{0}$ the neutral singleton — the right-handed neutrino is the Pascal zero with a job. The moment ladder adds $\mathrm{Tr}\, X^2 = 120 = 5!$ and $\mathrm{Tr}\, X^4 / \mathrm{Tr}\, X^2 = 2280/120 = 19$ (an E_8 exponent; audit). [I]

4.2 Carrier arithmetic (proved identities)

The discrete coefficients are not isolated numbers; they are exact carrier traces:

$$10\, b_1 = 41 = g_{\rm car}^2 + (g_{\rm car} - 1)^2 = \Omega_{\rm adm}\, \gamma_{\rm car} + 1 = 1 + h_{E_8} \tfrac{\dim S^+}{\dim \mathfrak{g}_{\rm SM}} = 1 + 30 \cdot \tfrac{16}{12} = 40 + 1, \quad [\mathrm{I}] \tag{20}$$

$$I_1^{\rm EW} = \frac{41}{48} = \mathrm{Var}_{S^+}(Y) \cdot b_1 = \frac{5}{24} \cdot \frac{41}{10} = \gamma_{\rm car} + \frac{1}{\Omega_{\rm adm}}, \quad [\mathrm{I}] \tag{21}$$

$$\mathrm{Tr}_{S^+}(X^2) = 120 = 5! \quad (X = 6Y), \qquad \delta_2 = \tfrac{5}{4} \delta_{\rm top}^2 = 2880\, c_3^8 = 4! \cdot 5!\, c_3^8 = 4!\ \mathrm{Tr}_{S^+}(X^2)\, c_3^8, \quad [\mathrm{I}] \tag{22}$$

$$\frac{5}{4} = \underbrace{\tfrac{3}{2} \cdot \tfrac{5}{6}}_{B\gamma} = \underbrace{\tfrac{D_{\rm start}}{\Omega_{\rm adm}}}_{60/48} = \underbrace{\tfrac{g_{\rm car}}{g_{\rm car}-1}}_{5/4} = \frac{\delta_2}{\delta_{\rm top}^2}. \quad [\mathrm{I}] \tag{23}$$

So: b_1 is a Pythagorean carrier identity (5^2+4^2); equivalently the anchor-product form $10b_1 = p_1p_3 + (g_{\rm car} - p_1) = 4\cdot 10 + (5-4) = 41$ reads “all four glue corners times all ten carrier pairs, plus the one leftover Higgs slot”; the electroweak exponent $41/48$ is the hypercharge variance times the $U(1)$ budget; the second seam defect $2880\,c_3^8$ is a product of factorial traces $4!\cdot 5!$; and the universal compression $5/4 = g_{\rm car}/(g_{\rm car}-1)$ recurs whenever the five-slot carrier projects onto four effective directions. As a minor curiosity, the SU(5)-tree weak mixing angle rewrites as $\sin^2\theta_W^{\rm tree} = \frac{3}{8} = N_{\rm fam}/(g_{\rm car}+N_{\rm fam}) = N_{\rm fam}/\operatorname{rank}(E_8)$ (the value is the standard one; the rewrite is a coincidence to note, not a claim). [P]

5 Grammar II — the seed grammar (one decoder u)

The Paper-3 seed is, exactly, a pure function of c_3:

$$\boxed{u = \varphi_0^{\rm ret} = \frac{4}{3}\,c_3 + \Omega_{\rm adm}\,c_3^4 = \underbrace{\frac{1}{6\pi}}_{\varphi_{\rm base}} + \underbrace{\frac{3}{256\pi^4}}_{\delta_{\rm top}} = 0.0531719521768} \qquad [\mathrm{I}]. \tag{24}$$

Its two coefficients are themselves carrier data: $\frac{4}{3} = \dim S^+/\dim\mathfrak{g}_{\rm SM} = 16/12 = 8/(g_{\rm car}+1)$ (so $\varphi_{\rm base} = 1/((g_{\rm car}+1)\pi)$) and $48 = \Omega_{\rm adm}$. The seed is thus a pure function of c_3 and the carrier counts.

One seed, four readouts (and their inverses) [I]

$$\lambda_C^2 = u(1-u), \quad \beta_{\rm rad} = \frac{u}{4\pi}, \quad \Omega_b = \Big(1-\frac{1}{4\pi}\Big)u, \quad \sin^2\theta_{13} = e^{-5/6}\,u,$$

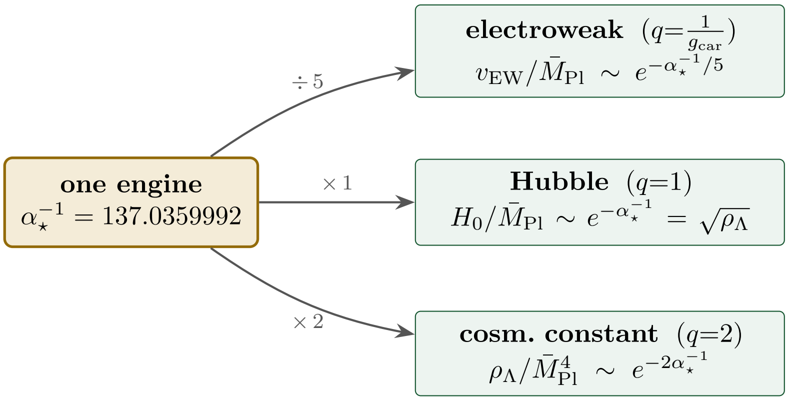
$$u = \frac{1-\sqrt{1-4\lambda_C^2}}{2} = 4\pi\,\beta_{\rm rad} = \frac{\Omega_b}{1-1/(4\pi)} = e^{5/6}\sin^2\theta_{13}$$

$$= \sum_{n\ge 0} C_n\,\lambda_C^{2n+2} = \lambda_C^2 + \lambda_C^4 + 2\lambda_C^6 + 5\lambda_C^8 + \cdots$$

One knob u turns four dials simultaneously: Cabibbo mixing (the Bernoulli width $\sqrt{u(1-u)}$), cosmic birefringence, the baryon fraction, and the reactor angle. Turning one forces the others; there is no free per-observable tuning. The exponent $5/6$ is the carrier trace $\gamma_{\rm car} = \mathrm{Tr}_E\,Y^2$, *not* the Euler–Mascheroni constant $0.5772\ldots$ (audit [A]).

6 Grammar III — the action grammar (exponentials of $\alpha_\star^{-1}$)

The relation in plain words. The fine-structure constant is not just “a number near 137”: it is the *one exponential engine* behind three wildly separated scales. Take the same $\alpha_\star^{-1} = 137.036$ and divide the carrier into it: $\frac{1}{5}$ of it sets the *electroweak* scale, 1× it sets the *Hubble* scale, 2× it sets the *cosmological constant*. The three action charges are the Pascal row $1:5:10$ of the five-slot carrier, and the Hubble rung is just the square root of the vacuum energy. So one constant fixes the electroweak hierarchy, dark energy and the expansion rate at once.



action charges $1 : g_{\rm car} : 2g_{\rm car} = 1 : 5 : 10 = \binom{5}{0} : \binom{5}{1} : \binom{5}{2}$ (Pascal row of K_5)

Read every dimensionless ratio through its *action* and ask for the action charge:

$$\boxed{x = R_x\, e^{-A_x}, \qquad A_x = q_x\, \alpha_\star^{-1} + \Delta_x, \qquad q_x \in \left\{\frac{1}{g_{\rm car}},\, 1,\, 2\right\}.} \tag{25}$$

The robust ladder and its residues are

sector	action A_x	charge q_x	residue R_x	value of $-\ln x$
electroweak $v_{\rm geo}/\bar M_{\rm Pl}$	$(\alpha_\star^{-1} + \delta_{\rm ph})/g_{\rm car}$	$1/g_{\rm car}$	$g_{\rm car}\beta_{\rm rad}^2$	36.85
Hubble $H_0/\bar M_{\rm Pl}$	$\alpha_\star^{-1}$	1	$1/(2\pi\sqrt{\Omega_\Lambda})$	138.68
cosm. const. $\rho_\Lambda/\bar M_{\rm Pl}^4$	$2\alpha_\star^{-1}$	2	$3/(4\pi^2)$	276.65

$$A_{\rm EW} = \frac{\alpha_\star^{-1} + \delta_{\rm ph}}{g_{\rm car}} = 27.5259, \qquad \mathcal{A}_\Lambda = 2\alpha_\star^{-1} = 274.072, \qquad \boxed{\mathcal{A}_{\rm EW}{:}\mathcal{A}_H{:}\mathcal{A}_\Lambda = 1{:}5{:}10}. \tag{26}$$

The leading ratio is exact: $\mathcal{A}_\Lambda/(\alpha_\star^{-1}/g_{\rm car}) = 2g_{\rm car} = 10$ (with the transport correction $\mathcal{A}_\Lambda/A_{\rm EW} = 9.957$). The claim is that the *action charges*, not the raw logarithms, form the ladder.

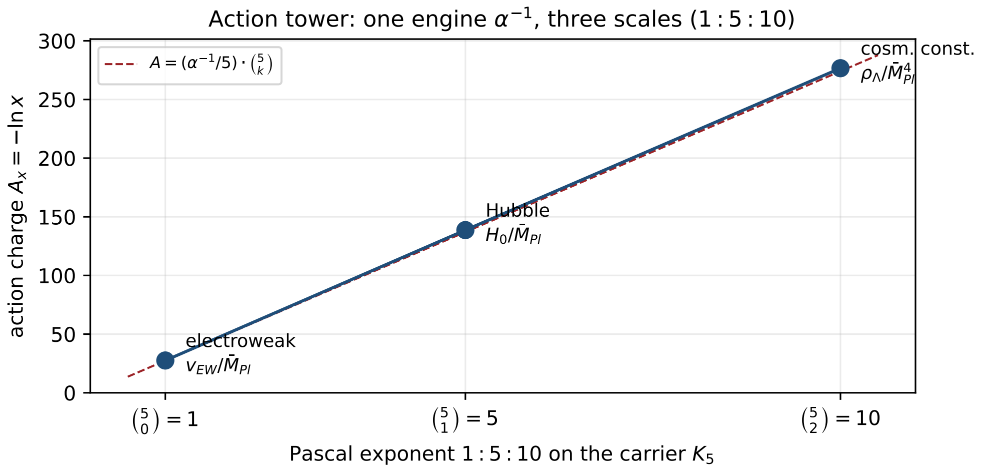


Figure 6: Action tower (data plot, `verification/make_figures.py`). The three action charges $A_x = -\ln x$ for the electroweak, Hubble and cosmological-constant ratios fall on the line $A = (\alpha_\star^{-1}/5)\binom{5}{k}$: one engine $\alpha_\star^{-1}$ generates all three scales with the Pascal exponents $1:5:10$.

6.1 The EM closure: $\alpha_\star^{-1}$ as an explicit fixed point (Paper 3, made explicit here)

The engine $\alpha_\star^{-1}$ is not an input — it is the unique root of an explicit closure equation (Paper 3, Thm. "exact electromagnetic closure"). With $c_3 = 1/(8\pi)$, $\varphi_{\text{base}} = 1/(6\pi)$, $q(\alpha) = \delta_{\text{top}} e^{-2\alpha}$ and the seam response $\varphi_{\text{seam}}(\alpha) = \varphi_{\text{base}} + q(\alpha)(1-q(\alpha))^{-5/4}$,

$$\boxed{F_{U(1)}(\alpha) = \alpha^3 - 2c_3^3\alpha^2 - \tfrac{4}{5}c_3^6\Big(\textstyle\sum_{f,j} L_{f,j} + N_\Phi\Big)\log\frac{1}{\varphi_{\text{seam}}(\alpha)} = 0} \qquad \text{[I]}, \tag{27}$$

where the coefficient is $\frac{4}{5}(\sum L + N_\Phi)c_3^6 = \frac{4}{5}\cdot 41\, c_3^6 = 8b_1c_3^6$, so the fine-structure constant reads the *flavor transport budget* $\sum L = 40$ and the $U(1)$ index $b_1 = 41/10$.

Lemma 1 (Existence and uniqueness of the EM fixed point)**.** $F_{U(1)}$ has *exactly one* positive root, and it is simple.

Proof. Write $F_{U(1)}(\alpha) = h(\alpha) - C\,g(\alpha)$ with $h(\alpha) = \alpha^3 - 2c_3^3\alpha^2$, $C = \frac{4}{5}c_3^6 M > 0$ ($M = 41$), and $g(\alpha) = -\log\varphi_{\text{seam}}(\alpha)$. Since $q(\alpha) = \delta_{\text{top}}e^{-2\alpha} \in (0,1)$ is strictly decreasing, $\varphi_{\text{seam}} = \varphi_{\text{base}} + q(1-q)^{-5/4}$ is C^1 and decreasing, so g is C^1, bounded, and slowly increasing with $g'(\alpha) = O(q)$ tiny; numerically $g \in [2.9342, 2.9343]$ on the admissible window. The cubic h has its interior critical point at $\alpha_c = \frac{4}{3}c_3^3 = 8.399\times 10^{-5}$, with $h(0) = 0$ and $h < 0$ on $(0,\alpha_c)$. *Monotonicity (corrected endpoint).* Since $F'_{U(1)}(\alpha) = 3\alpha^2 - 4c_3^3\alpha - C\,g'(\alpha)$ and at α_c the first two terms *cancel* ($3\alpha_c = 4c_3^3$), one has $F'_{U(1)}(\alpha_c) = -C\,g'(\alpha_c) \approx -5.89\times 10^{-10} < 0$: the derivative is *not* yet positive at α_c. Its first zero lies slightly above, at $\alpha_0 = 8.6264\times 10^{-5}$ (where $3\alpha_0^2 - 4c_3^3\alpha_0 = C\,g'(\alpha_0)$), with $F'_{U(1)} < 0$ on (α_c, α_0) and $F'_{U(1)} > 0$ on (α_0,∞). Now $F_{U(1)}(0) = -C\,g(0) < 0$, and $F_{U(1)}$ *decreases* on $(0,\alpha_0]$ (both h and $-Cg$ decrease on $(0,\alpha_c]$, then $F'_{U(1)} < 0$ on $[\alpha_c,\alpha_0]$), so $F_{U(1)}(\alpha_0) \approx -3.82\times 10^{-7} < 0$; thereafter $F_{U(1)}$ is strictly increasing to $+\infty$. Hence $F_{U(1)} < 0$ on $(0,\alpha_0]$ and strictly increasing on $[\alpha_0,\infty)$, so it crosses zero *exactly once*. The crossing has $F'_{U(1)}(\alpha_\star) = 1.58\times 10^{-4} > 0$, so the root is simple. A sign scan on $(0, 0.05)$ confirms a single change. *Rigorous version:* an **interval-arithmetic** enclosure (`mpmath.iv`) brackets the root in $[0.00729735256220970, 0.00729735256221000]$ with $F_{U(1)} < 0$ on the lower and > 0 on the upper endpoint (definite signs), and a 200-cell interval partition of $(\alpha_c, 0.05)$ contains exactly *one* indeterminate (root-bearing) band — a verified uniqueness, not merely a sampled one. [`verification/v3_em_alpha.py`] ☐

Solving:

$$\boxed{\alpha_\star = 0.00729735256220985\ldots, \qquad \alpha_\star^{-1} = 137.0359992168407\ldots} \quad \text{[I]} \tag{28}$$

[`verification/v3_em_alpha.py`] versus CODATA-2022 $\alpha_\star^{-1} = 137.035999177(21)$ — a relative deviation 2.9×10^{-10} ($\approx 1.9\sigma$ on the tiny experimental error; the older 2018 value 137.035999084 gave 9.7×10^{-10}, so the agreement *improved* in relative terms). The exponent $-5/4$ in φ_{seam} is the carrier discriminant norm $q(D_5)$, and $\delta_{\text{top}} = 48c_3^4$, so the closure is built only from c_3, the carrier, and the flavor budget. This discharges the former open audit point: the EM closure is a stated, existence-/uniqueness-proven, numerically reproduced equation, not a fit.

Why *this* function: the boundary $U(1)$ Ward reading (Theorem C) [I]/[P]

$F_{U(1)}$ is not a chosen ansatz but the reduced Ward identity of the boundary $U(1)$ partition function $Z_\partial = Z_{\text{Maxwell}}\,Z_{\text{Calderón}}\,Z_{\text{transport}}$, with three terms whose coefficients are compiler atoms, not fits:

$$\underbrace{\alpha^3}_{\text{Maxwell moment}} \quad \underbrace{-\,2c_3^3\alpha^2}_{\text{Calderón sheet }(2=|\mathbb{Z}_2|)} \quad \underbrace{-\,8b_1c_3^6\log\frac{1}{\varphi_{\text{seam}}}}_{C_6\text{ transport det.}} = 0, \qquad 8b_1 = \tfrac{4}{5}M = \tfrac{4}{5}(\textstyle\sum L + N_\Phi) = \tfrac{164}{5}.$$

The Calderón factor $2 = |\mathbb{Z}_2|$ (sheet), the transport power c_3^6 is the C_6 hexagon, and the seam exponent $-\frac{5}{4} = -q(D_5)$ is the carrier discriminant norm. **Typing:** the three-term decomposition and the coefficient identities ($8b_1 = \frac{4}{5}M$, $-\frac{5}{4} = q(D_5)$) are exact [I]; the *physical origin* as a boundary Ward closure (the lemmas $\partial_\tau \log Z_{\text{Maxwell}} = \alpha^3$, $\partial_\tau \log Z_{\text{Calderón}} = -2c_3^3\alpha^2$, $\partial_\tau \log \det_\zeta(1 - \mathcal{T}_{C_6,U(1)}) = -8b_1c_3^6 \log \frac{1}{\varphi_{\text{seam}}}$) is a conjectured [P] interpretation, not yet machine-proven. [`verification/v48_em_ward.py`]

The full ablation — which inputs are load-bearing and which only set the last digit — is in the Ablation appendix below.

Inversion test (this round). Running the closure backwards — inserting the *measured* α and solving for the integer budget $M = \sum L + N_\Phi$ — returns $M = 41.0000001\ldots$, i.e. the external measurement reconstructs the internal discrete budget $41 = 40 + 1$ to 10^{-7}. This is stronger than "we hit α": the equation reads the discrete 41 out of nature. [I]

6.2 The decuple bridge (new, exact headline)

Eliminating $\alpha_\star^{-1} = g_{\text{car}}\, A_{\text{EW}} - \delta_{\text{ph}}$ between the electroweak and the cosmological-constant lines gives a direct relation between the two scales:

Decuple bridge: Λ is the tenth power of the stripped EW scale [I]

$$\boxed{\frac{\rho_\Lambda}{\bar{M}_{\text{Pl}}^4} = \frac{3}{4\pi^2}\, e^{2\delta_{\text{ph}}} \left[\frac{v_{\text{geo}}}{5\,\beta_{\text{rad}}^2\,\bar{M}_{\text{Pl}}}\right]^{10}} \tag{29}$$

where the exponent $10 = 2g_{\text{car}}$ and $5 = g_{\text{car}}$. Verified to a relative residual $\sim 10^{-159}$.

Proof. From $v_{\text{geo}}/\bar{M}_{\text{Pl}} = g_{\text{car}}\beta_{\text{rad}}^2 e^{-A_{\text{EW}}}$, $e^{-A_{\text{EW}}} = (v_{\text{geo}}/\bar{M}_{\text{Pl}})/(g_{\text{car}}\beta_{\text{rad}}^2)$. From $\rho_\Lambda/\bar{M}_{\text{Pl}}^4 = \frac{3}{4\pi^2}e^{-2\alpha_\star^{-1}}$ and $\alpha_\star^{-1} = g_{\text{car}}A_{\text{EW}} - \delta_{\text{ph}}$, $e^{-2\alpha_\star^{-1}} = e^{2\delta_{\text{ph}}}e^{-2g_{\text{car}}A_{\text{EW}}} = e^{2\delta_{\text{ph}}}(e^{-A_{\text{EW}}})^{2g_{\text{car}}}$. Substituting and using $2g_{\text{car}} = 10$, $g_{\text{car}} = 5$ gives the boxed form. ☐

In words: the electroweak scale is not directly the Λ scale, but once the electroweak ratio is stripped of its small β_{rad} prefactor, Λ is its tenth power, and the "ten" is $2g_{\text{car}}$, not a fitted exponent. This is the direct algebraic compression of the action ladder.

6.3 The full power tower: x^1, x^5, x^{10} on the carrier graph K_5

The decuple bridge is one rung of a tower. Writing $x := e^{-A_{\text{EW}}} = v_{\text{geo}}/(5\beta_{\text{rad}}^2\bar{M}_{\text{Pl}})$ for the stripped electroweak unit, all three cosmological scales are powers of the *same* x with binomial exponents (verified exactly):

$$\boxed{\frac{v_{\text{geo}}}{5\beta_{\text{rad}}^2\bar{M}_{\text{Pl}}} = x^{\binom{5}{0}} = x, \qquad \frac{H_0}{\bar{M}_{\text{Pl}}} = x^{\binom{5}{1}}R_{\text{H}} = x^5 R_{\text{H}}, \qquad \frac{\rho_\Lambda}{\bar{M}_{\text{Pl}}^4} = x^{\binom{5}{2}}R_\Lambda = x^{10}R_\Lambda} \tag{30}$$

with residues $R_{\text{H}} = e^{\delta_{\text{ph}}}/(2\pi\sqrt{\Omega_\Lambda})$ and $R_\Lambda = \frac{3}{4\pi^2}e^{2\delta_{\text{ph}}}$. [I]. The exponents are the complete-graph counts of the five-slot carrier K_5:

$$H \leftrightarrow \text{vertices of } K_5 \ \left(\# = \tbinom{5}{1} = 5\right), \qquad \Lambda \leftrightarrow \text{edges of } K_5 \ \left(\# = \tbinom{5}{2} = 10\right).$$

So the cosmological constant is the product of one stripped-electroweak unit over every carrier *pair* (K_5 edge product), and the Hubble scale the product over every carrier *slot* (K_5 vertex product). This is the cleanest single-variable statement of the action ladder.

6.4 Hubble is the square root of Λ (corollary, exact)

On the comparison layer $\rho_\Lambda/\bar{M}_{\rm Pl}^4 = 3\Omega_\Lambda(H_0/\bar{M}_{\rm Pl})^2$; equating to the TFPT line gives

$$\boxed{\frac{H_0}{\bar{M}_{\rm Pl}} = \frac{e^{-\alpha_\star^{-1}}}{2\pi\sqrt{\Omega_\Lambda}}}, \qquad \mathcal{A}_H = -\ln\frac{H_0}{\bar{M}_{\rm Pl}} = \alpha_\star^{-1} + \ln\!\left(2\pi\sqrt{\Omega_\Lambda}\right) = 138.68. \tag{31}$$

So the Hubble rung is *not* an independent third hit: it is the square root of the vacuum energy plus a geometric cosmology factor. The genuine TFPT content is the single $e^{-\alpha_\star^{-1}}$. (The numerical coincidence $\mathcal{A}_H - \alpha_\star^{-1} \approx \pi^2/6$ is just $\ln(2\pi\sqrt{\Omega_\Lambda})$ and is Ω_Λ-dependent; not a claim. [A])

6.5 The ~ 123 orders of the cosmological constant

$$\log_{10}\frac{M_{\rm Pl}^4}{\Lambda_{\rm IR}} = \underbrace{\frac{2\alpha_\star^{-1}}{\ln 10}}_{119.028} + \underbrace{\log_{10}\frac{1}{\delta_{\rm top}}}_{3.920} = 122.948 \approx 123. \tag{32}$$

The "119" is the action $2\alpha_\star^{-1}$; the "≈ 4" is the seam defect $\delta_{\rm top} = \Omega_{\rm adm} c_3^4$.

High-precision cosmological constant: a $\sim 10^{-13}$ prediction [N]

Because $\alpha_\star^{-1}$ is a *derived* number (the EM fixed point, known to 13 digits) and the residue $3/(4\pi^2)$ is exact, the dimensionless vacuum energy is predicted essentially exactly:

$$\boxed{\frac{\rho_\Lambda}{\bar{M}_{\rm Pl}^4} = \frac{3}{4\pi^2}\, e^{-2\alpha_\star^{-1}} = 7.12533 \times 10^{-121}} \qquad \text{[N] [verification/v7\_gravity\_cosmo.py]}, \tag{33}$$

i.e. a dark-energy scale $\rho_\Lambda^{1/4} = \left(\frac{3}{4\pi^2}\right)^{1/4} \bar{M}_{\rm Pl}\, e^{-\alpha_\star^{-1}/2} = 2.23747$ meV (reduced Planck mass $\bar{M}_{\rm Pl} = 2.435323 \times 10^{18}$ GeV). The *dimensionless* ratio carries no observational input — its uncertainty is that of $\alpha_\star^{-1}$, $\sim 10^{-13}$ relative; the absolute meV value inherits only the Planck-mass uncertainty ($\sim 2 \times 10^{-5}$). Both are *far* sharper than the measured vacuum energy (Planck $\rho_\Lambda^{1/4} \simeq 2.24$ meV, known to $\sim 1\%$ and entangled with the H_0 tension). So TFPT pins the cosmological constant several orders of magnitude more precisely than it is currently measured — a genuine, falsifiable forward prediction: any future sub-percent ρ_Λ must land on 2.2375 meV.

6.6 The ladder is binomial; $2g_{\rm car} = \binom{g_{\rm car}}{2}$ singles out $g_{\rm car} = 5$

In units of the electroweak action $\alpha_\star^{-1}/g_{\rm car}$, the three rungs are the first three binomial coefficients of the five-slot carrier:

$$\frac{\mathcal{A}_{\rm EW}}{\alpha_\star^{-1}/g_{\rm car}} : \frac{\mathcal{A}_H}{\alpha_\star^{-1}/g_{\rm car}} : \frac{\mathcal{A}_\Lambda}{\alpha_\star^{-1}/g_{\rm car}} = 1 : g_{\rm car} : 2g_{\rm car} = \binom{g_{\rm car}}{0} : \binom{g_{\rm car}}{1} : \binom{g_{\rm car}}{2} = 1 : 5 : 10. \tag{34}$$

The first two are automatic ($\binom{g_{\rm car}}{0} = 1$, $\binom{g_{\rm car}}{1} = g_{\rm car}$). The non-trivial content is the cosmological-constant charge:

$$\boxed{2g_{\rm car} = \binom{g_{\rm car}}{2} \quad\Longleftrightarrow\quad g_{\rm car} = 5} \qquad \text{[I] [verification/v2\_carrier\_pascal.py]}, \tag{35}$$

since $\binom{g_{\rm car}}{2} = \frac{g_{\rm car}(g_{\rm car}-1)}{2} = 2g_{\rm car} \iff g_{\rm car} = 5$. So the "double overlap" charge $2g_{\rm car}$ equals the number of carrier-slot *pairs* $\binom{g_{\rm car}}{2}$ *only* at $g_{\rm car} = 5$: the cosmological constant is suppressed by one action unit per carrier pair, and this combinatorial reading is exclusive to the physical rank. It is an independent overdetermination of $g_{\rm car} = 5$ from the action grammar (cf. the landscape section).

6.7 Discriminant action theorem: the ladder reads the hypercharge discriminant

The $g_{\rm car}$ in the action ladder is not merely a rank: it is the square root of the discriminant of the carrier (hypercharge) polynomial. For a general split $b+s=g$ the Paper-2 polynomial is $bs\,Y^2+(s-b)Y-1=0$ with discriminant

$$\Delta_Y=(s-b)^2+4bs=(b+s)^2=g_{\rm car}^2,\qquad \text{so for } 6Y^2-Y-1=0:\ \Delta_Y=25,\ \sqrt{\Delta_Y}=5=g_{\rm car}. \tag{36}$$

Hence the action ladder is, exactly,

$$\boxed{\mathcal{A}_{\rm EW}=\frac{\alpha_\star^{-1}}{\sqrt{\Delta_Y}},\qquad \mathcal{A}_H=\sqrt{\Delta_Y}\,\mathcal{A}_{\rm EW},\qquad \mathcal{A}_\Lambda=\binom{\sqrt{\Delta_Y}}{2}\mathcal{A}_{\rm EW}}\qquad \text{[I]}. \tag{37}$$

The electroweak hierarchy therefore reads the *discriminant of the hypercharge polynomial*, tying the Standard-Model hypercharge derivation directly to the EW-hierarchy and Λ problems.

6.8 A two-dimensional action lattice

All scales sit on the integer lattice spanned by two generators — the gauge unit $\alpha_\star^{-1}/g_{\rm car}$ and the flavor unit $L_0=\ln(1/u)$ — as $S_x=a_x(\alpha_\star^{-1}/g_{\rm car})+b_xL_0+(\text{residue})$:

scale	a_x	b_x	axis
$v_{\rm geo}/\bar M_{\rm Pl}$	$1=\binom{5}{0}$	0	gauge
$H_0/\bar M_{\rm Pl}$	$5=\binom{5}{1}$	0	gauge
$\rho_\Lambda/\bar M_{\rm Pl}^4$	$10=\binom{5}{2}$	0	gauge
$m_e/\bar M_{\rm Pl}$	1	5	flavor
$m_\mu/\bar M_{\rm Pl}$	1	3	flavor
$m_\tau/\bar M_{\rm Pl}$	1	2	flavor

The cosmological/gauge sector runs along $b=0$ with binomial charges; the charged-fermion sector runs along $a=1$ with the integer flavor ladder $n_f=(5,3,2)$. This is the compact "periodic table" of TFPT scales: two integer labels per scale. [I] (charges) / [P] (residues).
A third (seam/gravity) generator completes the lattice: $L_c=\ln(1/c_3)=\ln(8\pi)$. The scalaron lives purely on it, $M_{\rm scal}/\bar M_{\rm Pl}=c_3^{7/2}=e^{-\frac{7}{2}L_c}$, and the seam defect is $\delta_{\rm top}=e^{-(4L_c-\ln 48)}$. So the three log-generators are $\{\alpha_\star^{-1}$ (hierarchy), $L_0=\ln(1/u)$ (flavor), $L_c=\ln(1/c_3)$ (seam/gravity)$\}$. [I]

7 The Einstein normaliser as a corrected tree value

Paper 5's gravitational normaliser $\xi=c_3/u$ has a transparent exact form. Writing $u=\frac{1}{6\pi}\big[1+\frac{9}{128\pi^3}\big]$,

$$\boxed{\xi=\frac{c_3}{u}=\frac{3}{4}\cdot\frac{1}{1+\dfrac{9}{128\pi^3}}=0.748303}\qquad \text{[I]}. \tag{38}$$

So $\xi_{\rm tree}=\frac{3}{4}=\dim\mathfrak{g}_{\rm SM}/\dim S^+=12/16$ is the exact leading value (the inverse of the $\frac{4}{3}=\dim S^+/\dim\mathfrak{g}_{\rm SM}$ family/gauge compression that sits in the seed $u_{\rm base}=\frac{4}{3}c_3$, so $\xi_{\rm tree}\,u_{\rm base}=c_3$), and the deviation is precisely the small topological seed correction $9/(128\pi^3)$, i.e. 0.226%. Gravity does not pick up a mysterious factor: first comes 3/4, then the seam defect shifts it minimally.

8 Λ-anchored metrology closure, and two anchors

Paper 5 leaves the absolute SI scale as a calibration. The dimensionless vacuum quotient lets one anchor it to the observed geometric curvature without circularity.

Theorem 1 (Λ-anchored metrology closure)**.** With $\lambda_\Lambda := \rho_\Lambda/\bar{M}_{\rm Pl}^4 = \frac{3}{4\pi^2}e^{-2\alpha_\star^{-1}}$ and the observed geometric curvature $\Lambda_{\rm geom}^{\rm obs} > 0$,

$$\bar{M}_{\rm Pl}^2 = \frac{\Lambda_{\rm geom}^{\rm obs}}{\lambda_\Lambda}, \qquad G_N^{\rm TFPT} = \frac{c^3}{8\pi\hbar}\frac{\lambda_\Lambda}{\Lambda_{\rm geom}^{\rm obs}} = \frac{3c^3}{32\pi^3\hbar\,\Lambda_{\rm geom}^{\rm obs}}\,e^{-2\alpha_\star^{-1}}.$$

Proof. $\bar{M}_{\rm Pl}^2 = 1/(8\pi G_N)$ and $\Lambda_{\rm geom} = \rho_\Lambda/\bar{M}_{\rm Pl}^2$; the branch sets $\rho_\Lambda = \lambda_\Lambda \bar{M}_{\rm Pl}^4$, so $\Lambda_{\rm geom} = \lambda_\Lambda \bar{M}_{\rm Pl}^2$, giving $\bar{M}_{\rm Pl}^2$ and then $G_N = \lambda_\Lambda/(8\pi\Lambda_{\rm geom}^{\rm obs})$; restoring $c^3/\hbar$ closes it. ☐

Non-circularity ([A], essential)

The anchor must be the curvature $\Lambda_{\rm geom}^{\rm obs}$ (units m^{-2}), *not* the SI vacuum energy density $\rho_\Lambda^{\rm obs} = \Lambda_{\rm geom}c^4/(8\pi G_N)$, which already contains G_N. The curvature follows from $\Lambda_{\rm geom}^{\rm obs} = 3\Omega_\Lambda H_0^2/c^2$.

Two independent anchors (crosscheck with teeth). The metrology can be pinned in two ways that should agree:

1. *Electroweak anchor* (Paper 5): the dimensionless $G_N v_{\rm phys}^2 = 4.06717 \times 10^{-34}$ fixes G_N once $v_{\rm phys}$ is calibrated non-gravitationally.
2. *Λ-curvature anchor* (this note): $G_N = \frac{3c^3}{32\pi^3\hbar\Lambda_{\rm geom}^{\rm obs}}e^{-2\alpha_\star^{-1}}$.

With the example anchor $H_0 = 67.36\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_\Lambda = 0.6847$ ($\Lambda_{\rm geom}^{\rm obs} = 1.08914 \times 10^{-52}\,\mathrm{m}^{-2}$) the Λ anchor gives

$$G_N^{\rm TFPT} = 6.65071 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}, \qquad G_N^{\rm TFPT}/G_N^{\rm CODATA} - 1 = -3.5 \times 10^{-3}. \tag{39}$$

Closure algebra [I]; SI anchor and numerical match [P] (the cosmological anchor is model dependent; the Hubble tension applies). EW crosscheck: the Λ-pinned $\bar{M}_{\rm Pl} = 2.43964 \times 10^{18}\,\mathrm{GeV}$ gives $v_{\rm geo} = 242.6\,\mathrm{GeV}$, needing $Z_{\rm EW} = (246.22/242.61)^2 = 1.030$.

Prefactor branch ([A])

This note uses $\Lambda_{\rm IR}/M_{\rm Pl}^4 = \delta_{\rm top}e^{-2\alpha_\star^{-1}}$. A competing prefactor $2c_3e^{-2\alpha_\star^{-1}}$ would differ by $2c_3/\delta_{\rm top} = 661.5$ and misscale G_N by the same factor; the branch must be versioned.

Mandatory Λ-layer typing ([A])

Three distinct Λ objects must never be conflated:

$$\underbrace{\frac{\rho_\Lambda}{\bar{M}_{\rm Pl}^4} = \frac{3}{4\pi^2}e^{-2\alpha_\star^{-1}}}_{\text{vacuum density quotient (dimensionless)}} \neq \underbrace{\frac{\Lambda_{\rm IR}}{M_{\rm Pl}^4} = \delta_{\rm top}e^{-2\alpha_\star^{-1}}}_{\text{seam-determinant / IR scale (dimensionless)}} \neq \underbrace{\Lambda_{\rm geom}^{\rm obs} = \frac{3\Omega_\Lambda H_0^2}{c^2}}_{\text{observed curvature }[\mathrm{m}^{-2}]}.$$

Only $\Lambda_{\rm geom}^{\rm obs}$ carries a dimension and is the non-circular anchor. Keeping the three typed protects the decuple bridge and the metrology closure from a spurious prefactor objection.

Measurement-near (physical) decuple bridge. Since $v_{\rm phys} = v_{\rm geo}\sqrt{Z_{\rm EW}}$ (Paper 5, charged-current residue, not a fit), the decuple bridge in the observable vev reads

$$\frac{\rho_\Lambda}{\bar{M}_{\rm Pl}^4} = \frac{3}{4\pi^2}e^{2\delta_{\rm ph}}\left[\frac{v_{\rm phys}}{5\beta_{\rm rad}^2\sqrt{Z_{\rm EW}}\,\bar{M}_{\rm Pl}}\right]^{10}. \tag{40}$$

This is the same identity, written in the low-energy charged-current scale rather than the geometric source scale; both forms should appear side by side. [I]/[P]

9 The landscape: $g_{\text{car}} = 5$ is overdetermined

Varying the carrier rank and rebuilding every constant from the same machinery (c_3 fixed), the EM-closure value $\alpha^{-1}(g_{\text{car}})$ is monotone and meets the observed value only at $g_{\text{car}} = 5$:

g_{car}	N_{fam}	$\dim S^+$	Ω_{adm}	$10b_1$	$\alpha^{-1}(g_{\text{car}})$	verdict
3	2	4	8	7	258.15	wrong α, wrong family count
4	2.5	8	20	17	187.31	non-integer families
5	**3**	**16**	**48**	**41**	**137.036**	**SM, 3 families, observed** α
6	3.5	32	112	97	101.30	non-integer families
7	4	64	256	225	75.61	wrong α

Several distinct requirements converge on $g_{\text{car}} = 5$: integer families (g_{car} odd); three families $(g_{\text{car}}+1)/2 = 3$; the observed α^{-1} on the monotone curve; the SM structure $(b, s) = (3, 2)$, $\dim S^+ = 16$; and $\Omega_{\text{adm}} = 48$, $10b_1 = 41$. A clean algebraic shadow: $\frac{g(g+1)}{2} = 2^{g-1} - 1$ has $g = 5$ as its non-trivial solution ([P]).

10 The E_8 carrier atlas — stated unambiguously

In the series E_8 is a *downstream scale grammar*, not a primitive cause: the boundary kernel fixes α, the carrier rank, families and admissibility without E_8 upstream. Three levels.

10.1 Level 1 — original E_8 data of the series ([I])

$$|R(E_8)| = 248 - 8 = 240 = 5 \cdot 48 = 5\,\Omega_{\text{adm}} = \operatorname{lcm}(\Omega_{\text{adm}}, D_{\text{start}}) = \operatorname{lcm}(48, 60), \tag{41}$$

$$D_{\text{start}} = g_{\text{car}} \cdot \dim \mathfrak{g}_{\text{SM}} = 5 \cdot 12 = 60, \qquad \gcd(\Omega_{\text{adm}}, D_{\text{start}}) = 12 = \dim \mathfrak{g}_{\text{SM}}, \tag{42}$$

$$\frac{|R(E_8)|}{D_{\text{start}}} = \frac{240}{60} = 4 = \frac{\Omega_{\text{adm}}}{\dim \mathfrak{g}_{\text{SM}}}, \qquad 2880 = D_{\text{start}}\,\Omega_{\text{adm}} = |R(E_8)| \cdot \dim \mathfrak{g}_{\text{SM}}, \tag{43}$$

$$\kappa_{E_8} = \frac{\gamma_{\text{car}}}{\ln(248/60)} = 0.58723, \qquad X(n, r, I_1) = \frac{\bar{M}_{\text{Pl}}}{8} \sin^2\theta_{13} \Big(\frac{60 - 2n}{60}\Big)^{\kappa_{E_8}} e^{-(8-r)/64} e^{-12\pi I_1}. \tag{44}$$

Coxeter/gauge typing of the lcm–gcd pair. Writing $D_{\text{start}} = 60 = 2h_{E_8}$ and $\Omega_{\text{adm}} = 48 = 6\,\operatorname{rank}(E_8)$, the relations become $\gcd(2h_{E_8}, 6r_{E_8}) = \dim \mathfrak{g}_{\text{SM}} = 12$ and $\operatorname{lcm}(2h_{E_8}, 6r_{E_8}) = |R(E_8)| = 240$: the local SM algebra is the gcd of the Coxeter double-period and the $\mathbb{Z}_6$ rank-occupancy, and the E_8 root count is their lcm. Likewise the EW exponent reads $I_1^{\text{EW}} = \frac{41}{48} = \frac{h_{E_8}}{36} + \frac{1}{6r_{E_8}} = \frac{5}{6} + \frac{1}{48}$ (Coxeter principal part plus a rank singlet). [I] (arithmetic) / [P] (typing).

10.2 Level 2 — new exact counting identities ([I])

$$|R(E_8)| = 240 = 16 \cdot 15 = \dim S^+\,(\dim S^+ - 1), \tag{45}$$

$$\dim E_8 = 248 = 8 \cdot 31 = (g_{\text{car}} + N_{\text{fam}})(2^{g_{\text{car}}} - 1), \tag{46}$$

$$|R(E_8)| = 8\,(31 - 1) = \operatorname{rank}(E_8)\,(2^{g_{\text{car}}} - 2), \qquad \operatorname{rank}(E_8) = 8 = g_{\text{car}} + N_{\text{fam}} = 5 + 3, \tag{47}$$

$$10\,b_1 = 41 = \frac{|R(E_8)|}{6} + 1, \qquad I_1^{\text{EW}} = \frac{41}{48} = \gamma_{\text{car}} + \frac{1}{\Omega_{\text{adm}}}, \tag{48}$$

with $31 = 2^{g_{\text{car}}} - 1$ the number of non-empty occupation words of the $[5, 4, 2]$ carrier code.

10.3 Level 2.5 — the even-flip transition atlas (a falsifiable program)

Identify S^+ with the even-weight binary code on five bits. A transition $x \to y \neq x$ has flip $F = x \oplus y$, even and nonempty, so $|F| \in \{2,4\}$; per state there are $\binom{5}{2} + \binom{5}{4} = 10 + 5 = 15$ flips, giving

$$240 = 16 \cdot (10+5) = \underbrace{160}_{|F|=2} + \underbrace{80}_{|F|=4}, \tag{49}$$

and the 120 undirected flips match the 120 E_8 root *pairs* ([I], counting). This makes the atlas conjecture *testable*: a genuine root map must send each flip to a norm-2 vector in $\mathbb{R}^8$ with inner products in $\{-2,-1,0,1,2\}$ reproducing the E_8 profile (per root: 1 self, 56 at $+1$, 126 orthogonal, 56 at -1, 1 anti).

Honest obstruction ([A])

The flip atlas splits $160+80$ by flip size; the standard E_8 (D_8) split is $112+128$. Since $160+80 \neq 112+128$, no flip-size-preserving map is an E_8 root system; a metrization, if it exists, must mix flip sizes. Until a norm-2, E_8-profile embedding is exhibited, E_8 stays a *counting* atlas, not a derived root origin.

Sharper two-stage audit (Coxeter before Weyl). A full root-system isomorphism is hard; a much earlier, cheaper test is the Coxeter spectrum. *Stage A:* find a natural order-30 operator C_{TFPT} on the even-flip atlas (or its 8-dimensional primitive-character space) with characteristic polynomial $\chi_{C_{\mathrm{TFPT}}}(t) = \Phi_{30}(t)$. *Stage B (only if A holds):* the norm-2, E_8-profile, Weyl-closure metrization. Stage A can fail or carry far sooner than B, so it is the right first falsifier. [A]

Audit only — *non-load-bearing* speculation: the $\dim 496$ heterotic count [A]

This box feeds no derivation; it is an arithmetic coincidence kept for completeness only, under the no-free-pattern rule. The even half S^+ and odd half S^- of the exterior algebra each have dimension $2^{g_{\mathrm{car}}-1} = 16$ and each carry $16 \cdot 15 = 240$ transitions. Together the full carrier ($2^{g_{\mathrm{car}}} = 32$ states) gives

$$\dim S^+(2^{g_{\mathrm{car}}} - 1) = 16 \cdot 31 = 496 = \dim(E_8 \times E_8) = \dim SO(32) = \binom{32}{2} \qquad \text{(count)},$$

i.e. the two halves read as two E_8 root atlases, all $\binom{32}{2} = 496$ pairs as $SO(32)$ — the two anomaly-free heterotic gauge dimensions. The *dimension* equality is exact; that the carrier algebra *is* either gauge group is unproven and not used anywhere. [A] (audit-only).

10.4 Level 2.7 — the carrier Coxeter theorem (the strongest new bridge)

Use the universal root-system identity $|R| = \mathrm{rank} \cdot h$ (Coxeter number h). With the carrier values $\mathrm{rank}(E_8) = g_{\mathrm{car}} + N_{\mathrm{fam}} = 8$ and $|R(E_8)| = \dim S^+(\dim S^+ - 1) = 240$,

$$\boxed{h_{E_8} = \frac{|R(E_8)|}{\mathrm{rank}(E_8)} = \frac{240}{8} = 30 = N_{\mathrm{fam}} \binom{g_{\mathrm{car}}}{2} = N_{\mathrm{fam}}\, A_\Lambda = 2 g_{\mathrm{car}} N_{\mathrm{fam}} = 2^{g_{\mathrm{car}}} - 2} \quad \text{[I]}. \tag{50}$$

The Coxeter number of E_8 is exactly three families times the cosmological pair action: each family contributes one Λ-decuple ($A_\Lambda = \binom{5}{2} = 10$) to the Coxeter period. Equivalently the action ladder is a Coxeter ladder,

$$A_H = \frac{h_{E_8}}{2N_{\mathrm{fam}}} = 5, \qquad A_\Lambda = \frac{h_{E_8}}{N_{\mathrm{fam}}} = 10, \tag{51}$$

so Hubble reads half a Coxeter period per family and Λ a full Coxeter period per family. [P]

Rank–Coxeter coupling. Equivalently $\dim S^+ = 2\,\mathrm{rank}(E_8) = 16$ and $h_{E_8} = 2(\dim S^+ - 1) = 2\cdot 15 = 30$, so $\dim E_8 = (g_{\mathrm{car}} + N_{\mathrm{fam}})(h+1) = 8\cdot 31$. The two counting forms $16\cdot 15 = 8\cdot 30 = 240$ coincide because $\dim S^+ = 2r$. Dropping ν^c sends $\dim S^+ \to 15$, which breaks *both* $\dim S^+ = 2r$ and $\dim S^+(\dim S^+ - 1) = 240$: ν^c is the state that closes the rank–Coxeter coupling, not an optional singlet. [I]/[P]

10.5 Level 2.8 — positive-root moment: $|R^+(E_8)| = \mathrm{Tr}_{S^+} X^2 = 5!$

The number of positive E_8 roots equals the master hypercharge moment:

$$\boxed{|R^+(E_8)| = 120 = \mathrm{Tr}_{S^+} X^2 = 5!} \qquad \text{[I]}, \tag{52}$$

so the second hypercharge moment of one family *is* the positive-root half-system. The whole integer chain then reads off $|R^+|$:

$$\sum_{f,j} L_{f,j} = \frac{|R^+(E_8)|}{3} = 40, \qquad 10b_1 = 1 + \frac{|R^+(E_8)|}{3} = 41, \qquad \delta_2 = 4!\,|R^+(E_8)|\,c_3^8. \tag{53}$$

The $\mathbb{Z}_6$ slice is then $|R(E_8)|/6 = \frac{|R^+|}{3} = 40 = \Omega_{\mathrm{adm}}\gamma$: one $\mathbb{Z}_6$ phase slice of the E_8 root atlas is exactly the full flavor transport budget. [I] (arithmetic) / [P] (interpretation).

10.6 Level 2.9 — the 2·3·5 Coxeter cyclotomic compiler (meta-theorem)

The three hard carrier atoms — weak rank 2, color rank / family count 3, carrier rank $g_{\mathrm{car}} = 5$ — generate the E_8 data *cyclotomically*, without taking E_8 as input. Set

$$\boxed{h := 2\cdot 3\cdot g_{\mathrm{car}} = 30, \qquad r := \varphi(h) = \varphi(30) = 8 = g_{\mathrm{car}} + N_{\mathrm{fam}},} \tag{54}$$

where h is the Coxeter number and $r = \varphi(2{\cdot}3{\cdot}5) = 8$ the rank. Then the standard $|R| = rh$, $\dim = r(h{+}1)$ give

$$|R(E_8)| = rh = 8\cdot 30 = 240, \qquad \dim E_8 = r(h+1) = 8\cdot 31 = 248, \qquad h+1 = 2^{g_{\mathrm{car}}} - 1 = 31. \tag{55}$$

Moreover the E_8 *exponents* are exactly the primitive residues modulo $h = 30$:

$$\mathrm{Exp}(E_8) = (\mathbb{Z}/30\mathbb{Z})^\times = \{1,7,11,13,17,19,23,29\}, \qquad \sum_{m\in(\mathbb{Z}/30)^\times} m = 120 = \mathrm{Tr}_{S^+} X^2 = |R^+(E_8)|, \tag{56}$$

so the second hypercharge moment equals the sum of the E_8 exponents and the positive-root count. The $\mathbb{Z}_6 \times \mathbb{Z}_5 \simeq \mathbb{Z}_{30}$ tact has $\varphi(30) = 8$ primitive characters, whose phases $e^{2\pi i m/30}$ are the Coxeter eigenphases of E_8; its characteristic polynomial is the 30th cyclotomic polynomial

$$\chi_C(t) = \Phi_{30}(t) = t^8 + t^7 - t^5 - t^4 - t^3 + t + 1. \tag{57}$$

Reading: E_8 here is not primarily a root count but the *Coxeter cyclotomic readout of the* $2, 3, 5$ *carrier*. [I] (arithmetic) / [P] (interpretation).

10.7 Level 3 — conjectural geometric readings ([P])

Conjecture 1 (Carrier transition atlas)**.** $R(E_8) \cong \{(a,b) \in S^+ \times S^+ : a \neq b\}$ as a structure-bearing transition set, since $16\cdot 15 = 240$. *Open:* a structure-preserving map (roots, lengths, Weyl data), not only the count.

Conjecture 2 (Least common refinement / rank-8 over the exhausted carrier)**.** $240 = \mathrm{lcm}(\Omega_{\mathrm{adm}}, D_{\mathrm{start}})$ reads E_8 as the least common refinement of the occupancy lattice and the $\mathbb{Z}_6$ phase lattice; $248 = (g_{\mathrm{car}} + N_{\mathrm{fam}})(2^{g_{\mathrm{car}}} - 1)$ reads it as a rank-8 structure over the 31 non-empty carrier words.

Two routes to E_8 — keep their status separate

Levels 1–2 are exact arithmetic. There are now *two* distinct routes to E_8, and they must not be conflated:

- **Glue route (closed, [L]):** $E_8 = (D_5 \oplus A_3) + \mu_4$-glue is built explicitly in Section 3 ($\operatorname{disc}(D_5) = \operatorname{disc}(A_3) = \mathbb{Z}_4$, glue index 4, 240 roots of norm 2, simple-root Gram det 1). This closes the former "blind 240-metrisation".
- **Direct flip-atlas route (open, [A], optional bonus):** the even-flip transition set $240 = 160 + 80$ is *not* a flip-size-preserving E_8 map, since $160 + 80 \neq 112 + 128$. A direct structure-preserving metrisation of the flip graph remains open — a nice-to-have, not load-bearing, because the glue route already delivers E_8.

So E_8 is closed via the glue; only the direct flip atlas is open.

11 $\mathbb{Z}_6$ monodromy, flavor and CP readouts

A single $\mathbb{Z}_6$ motif recurs in the quotient $G_{\text{phys}} = (SU(3)\times SU(2)\times U(1)_Y)/\mathbb{Z}_6$, the transport cusps $\{1, (2/3)^6, (1/3)^6\}$, $\delta_{\text{ph}} = z_\star^{1/6}$, and the CP phases

$$\delta_{\text{CKM}}^{(0)} = \frac{\pi}{3}, \quad \delta_{\text{CKM}} = \frac{\pi}{3} + 3\lambda_C^2 = 1.19823\,\text{rad}, \quad \delta_{\text{CP}}^{\text{PMNS}} = \frac{4\pi}{3} = 240^\circ. \tag{58}$$

From the documented inputs $s_{12} = \lambda_C$, $s_{23} = u/(1+\lambda_C)$, $s_{13} = \lambda_C^3/3$ one gets the new derived readouts $|V_{td}| = 0.00908$, $|V_{ts}| = 0.04264$, $J_{\text{CKM}} = 3.33\times 10^{-5}$. On the neutrino side, with $\sin^2\theta_{13} = e^{-5/6}u = 0.02311$, $\sin^2\theta_{23} = 0.4557$, $\delta_{\text{CP}} = 4\pi/3$ and the solar candidate $\sin^2\theta_{12} = \frac{1}{3} - \frac{u}{2} = 0.30675$, a near-maximal $|J_{\text{CP}}^{\nu}| \approx 0.030$ (sharp readout for DUNE/Hyper-Kamiokande). [P]

11.1 CP as triple seed variance; a closed observable quadrilateral

The Cabibbo angle is the Bernoulli width of the seed. The *exact* CKM phase is the family holonomy of Paper 3, $\delta_{\text{CKM}} = \arg \operatorname{Tr} \Pi_\triangle(\nabla_F^\star)$; on the small-area branch this has the compact readout form (with an $O(\lambda_C^4)$ remainder)

$$\lambda_C = \sqrt{u(1-u)}, \qquad \boxed{\delta_{\text{CKM}} - \frac{\pi}{3} = N_{\text{fam}}\, u(1-u) = 3\lambda_C^2} \qquad \text{[I]/[P] (readout; exact = holonomy).} \tag{59}$$

The coefficient 3 is the family number N_{fam}: weak CP is the $\mathbb{Z}_6$ base angle $\pi/3$ plus N_{fam} times the seed (Bernoulli) variance. The $\mathbb{Z}_6$ base angle now has an A_3 root: since $\mathbb{Z}_6 = |R^+(A_3)| = 6$ (the positive-root hexagon of the family algebra), the CP phases are integer steps of the A_3 holonomy quantum $2\pi/|R^+(A_3)|$,

$$\boxed{\delta_{\text{CKM}} = \frac{2\pi}{|R^+(A_3)|} + N_{\text{fam}}\, u(1-u), \qquad \delta_{\text{CP}}^{\text{PMNS}} = 4\cdot\frac{2\pi}{|R^+(A_3)|} = \frac{4\pi}{3}} \qquad \text{[I]/[P].} \tag{60}$$

So weak CP $=$ (A_3 positive-root phase step) $+$ (family count $\times$ seed variance), while strong CP is the *determinant* sector, erased. The solar angle reads as a carrier root plus a half-seed, $\sin^2\theta_{12} = \frac{1}{3} - \frac{u}{2} = -y_- - \frac{u}{2}$ (since $-y_- = \frac{1}{3}$), so PMNS = carrier root + seed + A_3 holonomy phase. [P]Strong CP is killed by the determinant selector ($\arg\det M_u = \arg\det M_d = 0$, $\bar\theta = 0$), while weak CP survives as the seed-deformed $\mathbb{Z}_6$ holonomy: CP is not forbidden, only *determinant* CP is. [P] — the strong-CP closure lives on the admissible QFT branch and is *conditional* on the analytic hypotheses of Paper 4 (reflection positivity, temperedness, clustering, mass gap, OS reconstruction); its status is therefore weaker than the discrete lattice identities, even though the semantics “determinant CP forbidden, holonomy CP allowed” is robust. Since $u = \Omega_b + \beta_{\rm rad}$ exactly, eliminating the seed closes the four UV-shadow observables into one cycle:

$$\lambda_C^2 = (\Omega_b + \beta_{\rm rad})(1 - \Omega_b - \beta_{\rm rad}), \quad \sin^2\theta_{13} = e^{-5/6}(\Omega_b + \beta_{\rm rad}), \quad \Omega_b = \Big(1 - \tfrac{1}{4\pi}\Big) e^{5/6} \sin^2\theta_{13}. \tag{61}$$

One measured member of $\{\lambda_C, \theta_{13}, \Omega_b, \beta_{\rm rad}\}$ fixes the other three. [I] (the four rows remain operationally distinct readout classes).

12 Mass hierarchy as a hexagon with one extra turn

The Paper-3 transport length packets are

$$L_u = (7, 3, 0), \qquad L_d = (7, 5, 2), \qquad L_e = (8, 5, 3), \qquad \sum_{f,j} L_{f,j} = 40 = \Omega_{\rm adm}\,\gamma_{\rm car} = 10b_1 - 1. \tag{62}$$

Modulo 6 they lie on related triples of the C_6 hexagon; the first generation carries one extra full turn ($u, d:\ 1 + 6$; $e:\ 2 + 6$), costing $\lambda_Y^6 = (u(1-u))^3 \approx 1.28 \times 10^{-4}$ and explaining why the first generation is so light — “one turn more”, not a free fit. [P]*Flagged proximity ([**A**], not exact):* the full-turn cost is close to the topological seam defect, $\lambda_C^6/\delta_{\rm top} = 1.061$ ($\sim 6\%$), hinting that one flavor winding $\approx$ one seam-defect unit $\delta_{\rm top} = \Omega_{\rm adm} c_3^4$ — a structural smell, not a theorem. Writing the length matrix $L = R + W$ with $R = L \bmod 6$ the residue shadow and W the full-winding part,

$$L = \begin{pmatrix} 7 & 3 & 0 \\ 7 & 5 & 2 \\ 8 & 5 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 3 & 0 \\ 1 & 5 & 2 \\ 2 & 5 & 3 \end{pmatrix}}_{R} + 6 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{W}, \qquad \sum R = 22,\ \sum W = 18 = 3 \cdot 6,\ \sum L = 40,$$

and the column sums are $(22, 13, 5)$ with $S_1 = 22 = \sum R$ and $S_2 + S_3 = 18 = \sum W$. So the first generation is light because it stores the *entire* residue budget plus the three full C_6 windings; generations 2 and 3 are the winding complement. [I] (arithmetic).

12.1 Lepton flavor decuple: the charged-lepton determinant carries u^{10}

The Paper-3 source masses on the $v_{\rm geo}$ branch are $(\hat m_e, \hat m_\mu, \hat m_\tau) = \frac{v_{\rm geo}\pi}{\sqrt{2}}\big(\frac{16}{7}u^5, \frac{4}{3}u^3, \frac{7}{6}u^2\big)$. Their product is a pure decuple in the seed:

$$\boxed{\frac{\hat m_e \hat m_\mu \hat m_\tau}{(v_{\rm geo}\pi/\sqrt{2})^3} = \Big(\tfrac{16}{7} \cdot \tfrac{4}{3} \cdot \tfrac{7}{6}\Big)\, u^{5+3+2} = \frac{32}{9}\, u^{10}} \qquad \text{[I]}, \tag{63}$$

with $5 + 3 + 2 = g_{\rm car} + b + s = 10 = \binom{5}{2}$. So the charged-lepton Yukawa powers $(n_e, n_\mu, n_\tau) = (5, 3, 2)$ *are* the three carrier numbers $(g_{\rm car}, b, s)$, and the lepton determinant carries the same decuple (u^{10}) that the cosmological constant carries in the gauge sector ($A_\Lambda = 10$): Λ and the charged-lepton determinant are two realisations of the same degree-$\binom{5}{2}$ sector.

Sum–product reading. The lepton powers $(5, 3, 2)$ encode *both* decuple numbers at once: their sum is the Λ action, their product is the E_8 Coxeter number,

$$5 + 3 + 2 = 10 = A_\Lambda, \qquad 5 \cdot 3 \cdot 2 = 30 = h_{E_8} = 2 \cdot 3 \cdot 5, \tag{64}$$

and $(5, 3, 2)$ is the prime factorisation of $h_{E_8} = 30$. So $\det M_\ell$, ρ_Λ and the E_8 Coxeter period read the same carrier core additively ($= 10$) and multiplicatively ($= 30$). [I]/[P]

12.2 Why the top Yukawa is ≈ 1: the $L = 0$ anchor (Froggatt–Nielsen ladder)

The transport length L_f is a Froggatt–Nielsen-type charge with expansion parameter λ_C:

$$\boxed{y_f \simeq \lambda_C^{L_f} \times \Lambda_f \ (\Lambda_f = \mathcal{O}(1)), \qquad y_t \simeq 1 \text{ because } L_t = 0} \qquad \text{[I]/[P]}. \tag{65}$$

The top sits at $L_t = 0$ (the *unwound* carrier direction), so $y_t = \Lambda_t \simeq 1$ ($\sqrt{2}\, m_t/v = 0.991$); every other charged fermion is suppressed by integer powers of $\lambda_C = 0.2244$. Numerically (with $\mathcal{O}(1)$ sector prefactors $\Lambda_f \in [0.43, 1.07]$):

f	t	b	τ	c	μ	s	u, d	e
L_f	0	2	3	3	5	5	7	8
$y_f/\lambda_C^{L_f}$	0.99	0.48	0.90	0.65	1.07	0.94	$\sim$ 0.4–0.9	0.46

So the entire charged-fermion hierarchy is one λ_C-ladder anchored by the top at $L = 0$; the lightness of the first generation ($L = 7, 8$) and the heaviness of the top ($L = 0$) are the two ends of the same integer winding spectrum. This is the standard Froggatt–Nielsen mechanism with λ_C as the flavon ratio and L_f as the FN charge — a clean bridge between TFPT transport and an established flavor mechanism.

13 Inflation interface (Paper 6)

The documented scalaron is Starobinsky R^2; with the pivot $N_\star = 57.1$:

$$\begin{aligned} n_s &= 1 - \frac{2}{N_\star} = 0.96497, \qquad r = \frac{12}{N_\star^2} = 0.00368, \\ \frac{M_{\text{scal}}}{\bar{M}_{\text{Pl}}} &= \sqrt{8\pi}\, c_3^4 = c_3^{7/2} = (8\pi)^{-7/2} \ \Rightarrow\ M_{\text{scal}} \approx 3.06 \times 10^{13}\,\text{GeV}. \end{aligned} \tag{66}$$

n_s matches Planck; $r = 0.0037$ is the falsifiable target for LiteBIRD/CMB-S4.
Amplitude (status-clean). The scalaron mass is a pure seam power,

$$\boxed{\frac{M_{\text{scal}}^2}{\bar{M}_{\text{Pl}}^2} = c_3^{r-1} = c_3^{\Omega_{\text{adm}} - 10 b_1} = c_3^7} \qquad \text{[I]}, \tag{67}$$

an *exact* identity (the exponent $7 = 48 - 41$ is the post-flavor, post-Higgs occupancy deficit). Paper 6 reduces the geometric gravity action to an FRW R^2/scalaron branch in which M_{scal} *is* the Starobinsky mass M; the identification is therefore derived in that sector, not posited. With the standard large-N amplitude $A_s = \frac{N_\star^2}{24\pi^2}(M/\bar{M}_{\text{Pl}})^2$ this gives

$$A_s = \frac{N_\star^2\, c_3^7}{24\pi^2} = \frac{N_\star^2\, \delta_2}{8640\pi} = 2.17 \times 10^{-9} \qquad \text{[P]/[A]}, \tag{68}$$

matching Planck ($\simeq 2.1 \times 10^{-9}$); $\delta_2 = 2880\, c_3^8$ is the second seam defect. The seam-power form is exact [I]; the amplitude is then determined *within* the R^2 sector, the only residual being the geometric-gravity (Hodge) closure on which that sector rests [P]. *Convention note:* $A_s \propto N_\star^2$, so the quoted 2.17×10^{-9} uses the pivot $N_\star = 57.1$; the alternative seed pivot $N_\star = 3/u - 1 = 55.4$ gives 2.05×10^{-9}. A number should only be quoted with $N_\star$ fixed.

14 Further simple bridges

- **Dark-energy equation of state $w = -1$ (exact).** The TFPT Λ is a genuine vacuum constant $\rho_\Lambda = \frac{3}{4\pi^2} e^{-2\alpha_\star^{-1}} \bar{M}_{\text{Pl}}^4$, not a rolling field, so $w = -1$ identically (no quintessence). A robust measurement $w \neq -1$ would falsify the Λ-branch. [I] (structural).

- **Effective neutrino number $N_{\rm eff} \simeq 3$.** Three families ($N_{\rm fam} = 3$) give three light neutrinos and the standard $N_{\rm eff} \simeq 3.04$. [I].
- **Top Yukawa** $\simeq 1$ as the $L = 0$ anchor (above): a bridge to the Froggatt–Nielsen flavor mechanism.
- *Flagged (not clean):* $m_H/v = 0.509$ is near $\frac{1}{2}$ (1.7%); $m_W/m_Z = 0.881$ matches the running $\sqrt{1 - \sin^2\theta_W(M_Z)}$ (not the tree $\sqrt{5/8} = 0.79$). No clean closed bridge for the Higgs/W/Z masses yet. [A]

15 Closures that the architecture rests on

Five results close or sharpen the discrete core. They are *proved in this document set* — the $\varphi_0^{\rm ret}$-ladder, the H2 realisation and the strong-CP/dynamics gap in document 2 (`tfpt_2_standard_model`), the $E_6\times A_2$ shadow in document 3 (`tfpt_3_e8_audit_bootstrap`) — and are collected here because the architecture above depends on them.

1. **The $\varphi_0^{\rm ret}$-ladder (whole spectrum, one formula).** Every fermion mass is $\hat{m}_{f,j} = \frac{v_{\rm geo}}{\sqrt{2}}\lambda_Y^{L_{f,j}}\Lambda_{f,j}$ with $\lambda_Y = \sqrt{\varphi_0^{\rm ret}(1-\varphi_0^{\rm ret})}$, and the closed evaluation gives the universal form $\lambda_Y^L\Lambda = \pi\, c\,(\varphi_0^{\rm ret})^k$ — the nine Λ are hexagonal-resolvent outputs (not free numbers); the *charged-lepton* ones are exact, the *quark* ones are reduced to the $(U_{\rm wall})$ selector. The $\varphi_0^{\rm ret}$-order ladder is $t(0) > \{b, c, \tau\}(2) > \{s, \mu\}(3) > \{u, d\}(4) > e(5)$. [I] (leptons exact, quark ratios closed v49/v71) / [A] (absolute quark scale)
2. **H2 reduced via the Mehta–Seshadri framework.** Mehta–Seshadri supplies the equivalence (stable parabolic $\leftrightarrow$ irreducible unitary); the TFPT-specific step — "stable = transport-minimal" and the geodesic C_6 word length — is conditional on the unitarity (U) of $\nabla_F^{\star}$ (Paper 3) and the parabolic-degree$\to$ C_6 dictionary. H2 is the sharp realisation statement: the D_4-equivariant stable structure on $\mathcal{O}(-2)\oplus\mathcal{O}(-1)^2$ realises the branch with $\det R = h(D_5) = 8$, principal 2-minors $(2, 3, 5)$. [I] (given (U))
3. **Strong CP / dynamics via the explicit gap.** $\theta_{\rm eff} = 0$ needs only γ_5-Hermiticity + polar mass + sheet involution + reflection positivity (no mass gap). The full dynamics closes on the physical sector because the C_6 transport transfer operator has the *explicit* spectrum $\{(k/3)^6\}$, giving the mass gap $\Delta = 6\log\frac{3}{2} = 2.43$ in closed form $\Rightarrow$ OS reconstruction $\Rightarrow$ measure on the admissible sector. [I] (phys. sector)
4. **$E_6\times A_2$ flavor shadow.** The residue matrix sits inside an E_8 secondary branching: $248 = \|R\|_F^2 + \det R + 2(\mathbf{1}^\top Ra)N_{\rm fam} = 78 + 8 + 2\cdot 27\cdot 3$, with $\|R\|_F^2 = 78 = \dim E_6$, $\det R = 8 = \dim A_2$. The seven E_8 slices form an audit raster (atlas note). [A]
5. **SM completeness.** The discrete structure and dimensionless core are closed; the dimensionful EW/QCD masses $(m_W, m_Z, m_H, \sin^2\theta_W, \alpha_s)$ are schema-level via $\mathcal{R}_{\rm SM}$, with m_H resting on the closed UV quartic $\lambda_\Phi = 1/(16\pi^2)$ (near-criticality). The only genuinely open SM point is the *exact* solar angle θ_{12} (leading $1/N_{\rm fam} = \frac{1}{3}$). [P]

16 Status summary and open audit points

Statement	Status	Note
Dual-root dictionary (Y, cusps, $\gamma_{\rm car}$, $\varphi_{\rm base}$)	[I]	exact, from $\{1/2, -1/3\}$
$10b_1 = 5^2 + 4^2 = \Omega_{\rm adm}\gamma + 1$	[I]	Pythagorean carrier identity
$I_1^{\rm EW} = 41/48 = \mathrm{Var}(Y)\, b_1$	[I]	exact
$\delta_2 = 4!\cdot 5!\, c_3^8$, $\mathrm{Tr}\, X^2 = 5!$	[I]	factorial traces
$5/4 = g_{\rm car}/(g_{\rm car} - 1)$ universal compression	[I]	exact
$u = \frac{4}{3}c_3 + 48c_3^4$; one-decoder identities	[I]	exact

$\xi = \frac{3}{4}/(1 + 9/128\pi^3)$	**[I]**	corrected tree value
Ladder $A_{\mathrm{EW}} : \mathcal{A}_H : \mathcal{A}_\Lambda = 1 : g_{\mathrm{car}} : 2g_{\mathrm{car}}$	**[I]**/**[P]**	Hubble rung is a corollary
Ladder $= \binom{g_{\mathrm{car}}}{0} : \binom{g_{\mathrm{car}}}{1} : \binom{g_{\mathrm{car}}}{2}$; $2g_{\mathrm{car}} = \binom{g_{\mathrm{car}}}{2} \Leftrightarrow g_{\mathrm{car}} = 5$	**[I]**	new overdetermination of $g_{\mathrm{car}} = 5$
2D action lattice $S_x = a(\alpha_\star^{-1}/g_{\mathrm{car}}) + bL_0$	**[I]**/**[P]**	two integer labels per scale
Decuple bridge $\rho_\Lambda/\bar{M}_{\mathrm{Pl}}^4 = \frac{3}{4\pi^2} e^{2\delta_{\mathrm{ph}}}[v/(5\beta_{\mathrm{rad}}^2 \bar{M}_{\mathrm{Pl}})]^{10}$	**[I]**	exact, $10 = 2g_{\mathrm{car}} = \dim \Lambda^2 E$
Family = ladder = Pascal row $\{1, 5, 10\}$; $16 = 1{+}5{+}10$	**[I]**	central new identification
$g_{\mathrm{car}} = 5$ via $2^{g_{\mathrm{car}}-1} = \binom{g_{\mathrm{car}}}{0}{+}\binom{g_{\mathrm{car}}}{1}{+}\binom{g_{\mathrm{car}}}{2}$	**[I]**	Pascal closure (unique)
$\mathrm{Tr}_{S^+} X^2 = 5!$ generates $40, 41, 240, \delta_2$	**[I]**	master moment
$\lvert R(E_8)\rvert = 16 \cdot 5 \cdot 3$; $\dim E_8 = 240 + 8$	**[I]**	E_8 compression
Even-flip atlas $240 = 160 + 80$	**[I]**/**[A]**	E_8 metrization open ($\neq 112 + 128$)
$\delta_{\mathrm{CKM}} - \pi/3 = 3u(1 - u)$	**[I]**	CP = triple seed variance
Observable quadrilateral ($u = \Omega_b + \beta_{\mathrm{rad}}$)	**[I]**	closed UV-shadow cycle
Power tower $v{=}x^1$, $H{=}x^5 R_H$, $\Lambda{=}x^{10} R_\Lambda$	**[I]**	K_5 vertices/edges
$496 = 16 \cdot 31 = \dim(E_8{\times}E_8) = \dim SO(32)$	**[I]**/**[P]**	heterotic-dimension count
$A_{\mathrm{EW}} = \alpha_\star^{-1}/\sqrt{\Delta_Y}$, $\Delta_Y = (b{+}s)^2 = 25$	**[I]**	discriminant action theorem
$\hat{m}_e \hat{m}_\mu \hat{m}_\tau \propto \frac{32}{9} u^{10}$; $(5, 3, 2) = (g_{\mathrm{car}}, b, s)$	**[I]**	lepton flavor decuple
$M_{\mathrm{scal}}/\bar{M}_{\mathrm{Pl}} = c_3^{7/2}$	**[I]**	scalaron seam power
$\delta_{\mathrm{CKM}} - \pi/3 = N_{\mathrm{fam}}\, u(1 - u)$	**[I]**	weak CP = $N_{\mathrm{fam}} \times$ seed variance
$L = R + W$: $\sum R = 22$, $\sum W = 18$	**[I]**	first generation = residue store
$\lvert R(E_8)\rvert/6 = \sum L = 40$ ($\mathbb{Z}_6$ slice)	**[I]**	one $\mathbb{Z}_6$ slice = flavor budget
$h_{E_8} = \lvert R\rvert/\,\mathrm{rank} = 30 = N_{\mathrm{fam}} \binom{5}{2} = N_{\mathrm{fam}} A_\Lambda$	**[I]**/**[P]**	carrier Coxeter theorem
$\lvert R^+(E_8)\rvert = \mathrm{Tr}_{S^+} X^2 = 120 = 5!$	**[I]**	positive-root moment
common decuple $\binom{5}{2}{=}A_\Lambda{=}u^{5+3+2}{=}h_{E_8}/N_{\mathrm{fam}}{=}10$	**[I]**/**[P]**	one carrier pair-decuple
$\lambda_C^6/\delta_{\mathrm{top}} = 1.061$ (winding $\approx$ seam defect)	**[A]**	flagged proximity (not exact)
$y_f \simeq \lambda_C^{L_f}$, top $L_t = 0 \Rightarrow y_t \simeq 1$	**[I]**/**[P]**	Froggatt–Nielsen ladder
$w = -1$ (true Λ); $N_{\mathrm{eff}} \simeq 3$	**[I]**	dark energy / neutrino species
3-generator lattice $\{\alpha_\star^{-1}, L_0, L_c\}$; $M_{\mathrm{scal}} = e^{-\frac{7}{2} L_c}$	**[I]**	seam axis $L_c = \ln(1/c_3)$
$\dim S^+ = 2\,\mathrm{rank}(E_8)$, $h_{E_8} = 2(\dim S^+{-}1) = 2^g{-}2$	**[I]**	rank–Coxeter coupling (ν^c closes it)
$\gcd(2h, 6r) = \dim \mathfrak{g}_{\mathrm{SM}} = 12$, $\mathrm{lcm} = 240$	**[I]**/**[P]**	SM algebra = gcd; E_8 count = lcm
$I_1^{\mathrm{EW}} = h_{E_8}/36 + 1/(6r) = 41/48$	**[I]**/**[P]**	Coxeter part + rank singlet
$M_{\mathrm{scal}}^2/\bar{M}_{\mathrm{Pl}}^2 = c_3^{r-1} = c_3^7$ exact; $A_s = N_\star^2 c_3^7/24\pi^2$	**[I]** / **[P]**/**[A]**	seam power exact; A_s conditional (Starobinsky)
$h_{E_8} = 2{\cdot}3{\cdot}5 = 30$, $r = \varphi(30) = 8$	**[I]**/**[P]**	$2, 3, 5$ Coxeter cyclotomic compiler
$\mathrm{Exp}(E_8) = (\mathbb{Z}/30)^\times$, $\sum = 120$, $\chi_C = \Phi_{30}$	**[I]**/**[P]**	exponents = primitive residues
lepton $(5, 3, 2)$: $\sum{=}10{=}A_\Lambda$, $\prod{=}30{=}h_{E_8}$	**[I]**/**[P]**	sum=action, product=Coxeter
Hubble $H_0/\bar{M}_{\mathrm{Pl}} = e^{-\alpha_\star^{-1}}/(2\pi\sqrt{\Omega_\Lambda})$	**[I]**	exact corollary
Λ-anchored closure $G_N = \frac{3c^3}{32\pi^3 \hbar \Lambda_{\mathrm{geom}}} e^{-2\alpha_\star^{-1}}$	**[I]**/**[P]**	G_N to -0.35%
$g_{\mathrm{car}} = 5$ overdetermined	**[P]**	monotone $\alpha^{-1}(g_{\mathrm{car}})$
$\lvert R(E_8)\rvert = 240 = 16 \cdot 15$, $248 = 8 \cdot 31$	**[I]**	counting identities
$E_8 \leftrightarrow$ carrier-transition map	**[P]**	needs structure map
$\delta_{\mathrm{CP}} = 4\pi/3$, $\theta_{12} = \frac{1}{3} - \frac{u}{2}$	**[P]**	PMNS readouts
$\lvert V_{td}\rvert, \lvert V_{ts}\rvert, J_{\mathrm{CKM}}, J_{\mathrm{CP}}^\nu$; n_s, r	**[P]**	derived readouts

Open audit points: (1) *discharged* — the EM closure $F_{U(1)}(\alpha) = 0$ is now stated explicitly (Grammar III), with existence/uniqueness and root $\alpha_\star^{-1} = 137.0359992168$ (2.9×10^{-10}, 1.9σ vs CODATA-2022); (2) keep $\gamma_{\mathrm{car}} = 5/6$ vs γ_E distinct; (3) type M_{Pl} vs $\bar{M}_{\mathrm{Pl}}$ and "pure" ($\mathcal{A}_\Lambda = 274.07$) vs "total" ($-\ln(\delta_{\mathrm{top}} e^{-2\alpha_\star^{-1}}) = 283.10$, $-\ln(\rho_\Lambda/\bar{M}_{\mathrm{Pl}}^4) = 276.65$) actions consistently; (4) version the Λ prefactor branch; (5) the scalaron-mass→ Starobinsky-M identification is the *only* condition for $A_s = N_\star^2 c_3^7/(24\pi^2)$ (the seam power $M_{\mathrm{scal}}^2/\bar{M}_{\mathrm{Pl}}^2 = c_3^7$ itself is exact); (6) the E_8 *lattice* is now built explicitly as $(D_5 \oplus A_3) + \mu_4$-glue (Section 3), so the former "blind 240-metrisation" is closed. The flavor status (companion parabolic-weight note) now splits cleanly: the *anchor multiset* $\{1,1,2\}$ is derived (Schur–Horn + stability) [I]; the *sector assignment* $(1,1,2)$ follows from the D_5 cusp ordering [I]/[P]; the *first-generation* $+6$ *winding* is the hierarchy convention [P]; the six *non-anchor residues* are then forced (residue sets = D_6 orbit of $\{0,1,3\}$, minus anchor, monotone) [I]. The single remaining geometric input is (H2), the specific D_6 cusp labelling on $\mathbb{P}^1 \setminus \mu_4$ — sharply characterised by the residue-matrix spectral selector ($\det R = h(D_5)$, principal minors $(2,3,5)$), with a naive cusp-position shortcut explicitly ruled out [P]. The explicit Riemann–Hilbert construction confirms the rigid $SU(3)$ system and its A_3 Coxeter rotation; the genuine open step is the global parabolic-weight (Hodge) realisation of that selector.

Honest negatives (reported to keep the discipline visible). The TFPT lepton ladder gives a Koide ratio $Q = (\sum m_\ell)/(\sum \sqrt{m_\ell})^2 = 0.6645$, close to but *not* the empirical $2/3 = 0.66667$ (-0.33%); TFPT is Koide-*like* but does not reproduce the precise relation. The baryon asymmetry $\eta_B \approx 6 \times 10^{-10}$ and the proton/electron ratio $m_p/m_e = 1836$ do *not* fall on a clean action charge or L_0 ladder; they remain outside the present grammar. These are stated so the strong items are not read as a uniform success.

17 Consolidated picture

The three nested grammars

$$\begin{aligned} \textbf{Carrier:}\quad & 3+2 \Rightarrow \{y_-, y_+\} = \{-\tfrac{1}{3}, \tfrac{1}{2}\} \Rightarrow Y,\ \dim S^+ = 16,\ N_{\mathrm{fam}} = 3,\ \Omega_{\mathrm{adm}} = 48,\ b_1 = \tfrac{41}{10}, \\ \textbf{Seed:}\quad & u = \tfrac{4}{3}c_3 + 48c_3^4 \Rightarrow \{\lambda_C,\ \beta_{\mathrm{rad}},\ \Omega_b,\ \sin^2\theta_{13}\}, \\ \textbf{Action:}\quad & \alpha_\star^{-1} \Rightarrow A_{\mathrm{EW}} = \tfrac{\alpha_\star^{-1}}{g_{\mathrm{car}}},\ \mathcal{A}_\Lambda = 2\alpha_\star^{-1},\ \mathcal{A}_H = \alpha_\star^{-1} \Rightarrow \frac{\rho_\Lambda}{\bar{M}_{\mathrm{Pl}}^4} = \frac{3}{4\pi^2} e^{2\delta_{\mathrm{ph}}} \left[\frac{v_{\mathrm{geo}}}{5\beta_{\mathrm{rad}}^2 \bar{M}_{\mathrm{Pl}}}\right]^{10}. \end{aligned}$$

A small grammar generates many sectors: the Standard-Model packet, flavor, the electroweak hierarchy, the cosmological constant, the metrology, and the inflation/CMB interface. The strongest new statements are exact: the decuple bridge ties Λ to the tenth power of the stripped electroweak scale with $10 = 2g_{\mathrm{car}}$; the Hubble scale is the square root of Λ; and the Einstein normaliser is the corrected tree value $\frac{3}{4}/(1 + 9/128\pi^3)$. None of this is claimed as a proof of a final theory; it is offered as a tightening of the existing architecture for your assessment.

The compiler in one line

$$\boxed{\text{TFPT} = \text{even carrier code} + \text{seed decoder} + 2{\cdot}3{\cdot}5 \text{ Coxeter compiler}}$$

From the three atoms $(2, 3, g_{\mathrm{car}}{=}5)$:

$$h_{E_8} = 2{\cdot}3{\cdot}5 = 30, \qquad r_{E_8} = \varphi(30) = 8 = g_{\mathrm{car}}{+}N_{\mathrm{fam}},$$

$$|R(E_8)| = rh = 240, \qquad \dim E_8 = r(h{+}1) = 8{\cdot}31 = 248,$$

$$\mathrm{Exp}(E_8) = (\mathbb{Z}/30)^\times, \quad \sum \mathrm{Exp} = 120 = \mathrm{Tr}_{S^+} X^2 = |R^+|, \quad \chi_C = \Phi_{30},$$

$$\dim S^+ = 1{+}10{+}5 = 16 = 2r, \quad A_{\mathrm{EW}}{:}\mathcal{A}_H{:}\mathcal{A}_\Lambda = 1{:}5{:}10, \quad \mathcal{A}_H = \tfrac{h}{6},\ \mathcal{A}_\Lambda = \tfrac{h}{3},\ N_{\mathrm{fam}} = \tfrac{h}{\mathcal{A}_\Lambda} = 3,$$

$$\sum L = \frac{|R^+|}{3} = 40, \quad 10b_1 = 1 + \frac{|R^+|}{3} = 41 = 1 + h\frac{\dim S^+}{\dim \mathfrak{g}_{\mathrm{SM}}},$$

$$\det M_\ell \propto u^{10}, \qquad \frac{\rho_\Lambda}{\bar{M}_{\mathrm{Pl}}^4} \propto \left[\tfrac{v_{\mathrm{geo}}}{5\beta_{\mathrm{rad}}^2 \bar{M}_{\mathrm{Pl}}}\right]^{10}, \qquad A_{\mathrm{EW}} = \frac{\alpha_\star^{-1}}{\sqrt{\Delta_Y}}, \quad \sqrt{\Delta_Y} = g_{\mathrm{car}} = 5.$$

The common decuple ($= 10$): $\binom{g_{\mathrm{car}}}{2}$ (carrier pairs) $= A_\Lambda$ (cosmological action) $= u^{5+3+2}$ (lepton determinant) $= h_{E_8}/N_{\mathrm{fam}}$ (Coxeter per family). Λ, the charged-lepton determinant, the E_8 Coxeter structure and the action ladder all read the *same* carrier pair-decuple. The open core is one question: can the even-flip atlas be metrised as a genuine E_8 root system (the 160+80 vs 112+128 obstruction)?

Part II

The Coxeter–cyclotomic compiler and the explicit E_8 construction

Abstract

This short companion compresses the TFPT 4.5 bridge layer to a single discrete object. The claim is that the Standard-Model family, the cosmological action ladder, the E_8 counting data, the flavor budget, inflation and the cosmological constant are *not* independent coincidences but projections of one structure: the even-weight code C^+ on $g_{\text{car}} = 5$ slots, together with its $\mathbb{Z}_{30}$ Coxeter spectrum and the single seam seed $c_3 = 1/(8\pi)$. The decisive new result is that C^+ is precisely the half-spin weight orbit of $D_5 = \mathfrak{so}_{10}$ ($\mathrm{Aut}(C^+) = W(D_5)$, order 1920), that the family geometry $\mathbb{P}^1 \setminus \mu_4$ is an $A_3 = \mathfrak{su}_4$, and that E_8 is then the standard *closure algebra* of $D_5 \times A_3$ — not an input. The fusion is exact: $\mathrm{disc}(D_5) = \mathrm{disc}(A_3) = \mathbb{Z}_4$, so the four family corners μ_4 are precisely the glue code that closes $D_5 \oplus A_3$ into the unimodular E_8 lattice. The glue is metric: the discriminant-form norms $q(D_5) = \frac{5}{4}$ and $q(A_3) = \frac{3}{4}$ sum to the E_8 root norm 2 — and these are exactly the pre-existing TFPT constants $\delta_2/\delta_{\text{top}}^2 = \frac{5}{4}$ and $\xi_{\text{tree}} = \frac{3}{4}$. The A_3 side is explicit: the residue pairing on $H^1(\mathbb{P}^1 \setminus \mu_4)$ is the A_3 Cartan form, and the Paper-2 corner rotation $z \mapsto iz$ acts as the A_3 Coxeter element (order 4, spectrum $\{i, -1, -i\}$). The lift is no longer a count: the 240 E_8 roots are *explicitly* built from $D_5 \oplus A_3$ plus the spinor (μ_4) glue, all of norm 2, Cartan determinant 1. Seven statements carry it; every lattice identity below is checked exactly, and the analytic metric is computed and characterised (a D_4-equivariant deformation), so no *lattice* structural step is left open in the $D_5 \times A_3$ glue route (the upstream boundary kernel, EM closure and flavor Hecke saturation remain separate open problems). Master statement: *five carrier slots give D_5, four family corners give A_3, and μ_4 glues $D_5 \times A_3$ into the E_8 atlas.*

The compression

$$\boxed{\begin{aligned}&\mathrm{TFPT} = \underbrace{D_5 \text{ half-spin carrier}}_{C^+} + \underbrace{A_3 \text{ four-point family}}_{\mathbb{P}^1\setminus\mu_4} + c_3 \text{ seam seed},\\ &E_8 = \text{closure of } D_5 \times A_3\end{aligned}}$$

with $c_3 = \dfrac{1}{8\pi}$, $g_{\text{car}} = 5$, $h := 2 \cdot 3 \cdot g_{\text{car}} = 30$, $r := \varphi(h) = \varphi(30) = 8$, $N_{\text{fam}} = 3$, $N_\Phi = 1$, $\dim \mathfrak{g}_{\text{SM}} = 12$. Equivalently $C^+ + \mathbb{Z}_{30}$ Coxeter compiler $+\ c_3$ seed; the $D_5 \times A_3$ reading is the structural lift of the same object.

Theorem 1 — the even carrier code

Let E be the rank-$g_{\text{car}} = 5$ carrier, split $E = E_3 \oplus E_2$ (color $b = 3$, weak $s = 2$). The one-generation spinor packet is the even exterior algebra, equivalently the even-weight binary code

$$S^+ = \Lambda^0 E \oplus \Lambda^2 E \oplus \Lambda^4 E \;\cong\; C^+ = \{x \in \mathbb{F}_2^5 : |x| \text{ even}\}, \qquad \dim S^+ = |C^+| = 2^{g_{\text{car}}-1} = 16. \quad (69)$$

The two carrier roots $y_+ = \frac{1}{2}$, $y_- = -\frac{1}{3}$ generate all hypercharges ($Y(Q_L) = y_+ + y_- = \frac{1}{6}$, $Y(u^c) = 2y_-$, $Y(e^c) = 2y_+$, $Y(\nu^c) = 0$) and the transport cusps $\{1, \frac{2}{3}, \frac{1}{3}\} = \{2y_+, -2y_-, 2(y_+ + y_-)\}$, with $\gamma_{\text{car}} = \mathrm{Tr}_E\, Y^2 = 3y_-^2 + 2y_+^2 = \frac{5}{6}$. [I]

Gauge-rank factorisation (`[verification/v79_review_identities.py]`**).** The carrier polynomial factors into the two gauge ranks directly:

$$\boxed{6Y^2 - Y - 1 = (2Y-1)(3Y+1)}, \qquad Y = \tfrac{1}{2}\ (s{=}2,\ SU(2)), \quad Y = -\tfrac{1}{3}\ (b{=}3,\ SU(3)),$$

i.e. $(sY-1)(bY+1)$ — the polynomial splits into one linear representative of each Standard-Model non-abelian factor; the 5-slot carrier *forces* the split into $SU(3) \times SU(2)$.

Hypercharge Lucas–Binet resonances [I] (mode reading [P], [`verification/v79_review_identities.py`])

The integer carrier charges $(q_+, q_-) = (3, -2)$ (the primitive trace-free pair, $X^2 - X - 6 = (X-3)(X+2)$, Lean `unique_carrier_pair`) generate the Lucas U-sequence

$$D_n = \frac{3^n - (-2)^n}{5}: \qquad D_1, \ldots, D_6 = 1,\ 1,\ 7,\ 13,\ 55,\ 133,$$

which is, *in order*, $(N_\Phi,\ \cdot$, scalaron exponent 7, $\Delta_Q{=}13$, quark numerator 55, $\dim E_7{=}133)$. In particular the quark ratio reads $c_u/c_d = D_5/(N_{\text{fam}}^2 D_4) = \frac{55}{117}$ — a single recurrence of the hypercharge generator, not a free integer. [I] for the arithmetic; the per-mode physical identification is a [P] cross-domain reading (only $n \le 6$, the carrier-slot range, is meaningful).

Theorem 2 — Exterior–Action Duality

On C^+ the flip $F = x \oplus y$ is even, so $|F| \in \{0, 2, 4\}$. The three relations R_0, R_2, R_4 form a symmetric association scheme (the even Hamming scheme on five bits) with

$$\boxed{\begin{array}{l}\text{valencies } (1, 10, 5) = (\dim \Lambda^0 E, \dim \Lambda^2 E, \dim \Lambda^4 E), \\ \text{spectral multiplicities } (1, 5, 10) = A_{\text{EW}} {:} \mathcal{A}_H {:} \mathcal{A}_\Lambda\end{array}} \qquad \text{[I]}. \tag{70}$$

The eigenvalues (computed directly) are $A_2 : \{10^{(1)}, 2^{(5)}, (-2)^{(10)}\}$ and $A_4 : \{5^{(1)}, (-3)^{(5)}, 1^{(10)}\}$, with $A_2 + A_4 = K_{16} - \mathbf{1}$. The two graphs are named objects:

$$\begin{aligned} &R_2(C^+) = \text{folded/complement, SRG}(16, 10, 6, 6) \text{ (the Clebsch complement);} \\ &\boxed{R_4(C^+) = \text{Clebsch graph, SRG}(16, 5, 0, 2)} \qquad \text{[I]}. \end{aligned} \tag{71}$$

Remark. The Standard-Model family is the *valency* side $(1, 10, 5)$ of one scheme; the cosmological action ladder is its *spectral* side $(1, 5, 10)$. The action ladder is therefore not merely "also binomial" — it is the spectral dual of the even exterior family. [P]

This duality is the MacWilliams transform ([`verification/v79_review_identities.py`]). C^+ is the even-weight code $[5, 4, 2]$; its dual $(C^+)^\perp$ is the repetition code $[5, 1, 5] = \{00000, 11111\}$ (the uniform/vacuum word). The MacWilliams identity sends the weight enumerator of the dual to that of C^+, namely $(1, 10, 5)$ — so the valency↔spectral duality *is* the coding-theoretic UV/IR duality, and the carrier grading $\Lambda^{\text{even}}(5) = 1{+}10{+}5$ is the MacWilliams image of the repetition vacuum. [L] (code identity) / [P] (holographic reading).

The full Bose–Mesner algebra (machine-checkable) fixes the duality beyond eigenvalues:

$$A_2^2 = 10\,\mathbf{1} + 6A_2 + 6A_4, \quad A_2 A_4 = 3A_2 + 4A_4, \quad A_4^2 = 5\,\mathbf{1} + 2A_2, \tag{72}$$

with eigenmatrix and multiplicity orthogonality

$$P = \begin{pmatrix} 1 & 10 & 5 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{pmatrix}, \quad \det P = -64, \qquad \boxed{P^\top \operatorname{diag}(1, 5, 10)\, P = \operatorname{diag}(16, 160, 80)} \qquad \text{[I]}, \tag{73}$$

so the flip split $240 = 160{+}80$ ($|F| = 2, 4$) appears directly in the scheme's eigenmatrix orthogonality.

Carrier $= D_5$ (verified). The 16 vertices of C^+ are exactly the weights of the $D_5 = \mathfrak{so}_{10}$ half-spinor $\frac{1}{2}(\pm1, \ldots, \pm1)$ with an even number of minus signs. The flip-2 and flip-4 graphs are translation- and permutation-invariant, so the *affine* group $C^+ \rtimes S_5$ (translations $\rtimes$ coordinate permutations) acts by graph/scheme automorphisms; the group closure gives the automorphism group of the even Hamming scheme (equivalently the Clebsch graph),

$$\boxed{\mathrm{Aut}_{\mathrm{aff}}(C^+) = C^+ \rtimes S_5, \qquad |\,\mathrm{Aut}_{\mathrm{aff}}(C^+)| = 2^{g_{\mathrm{car}}-1}\, g_{\mathrm{car}}! = 16 \cdot 120 = 1920 = |W(D_5)|} \quad [\mathrm{I}], \tag{74}$$

and $1920 = r\,|R(E_8)| = 8 \cdot 240$. (The *linear* code automorphism group alone is only the coordinate group S_5 of order 120; the factor $2^{g_{\mathrm{car}}-1} = 16$ is the translation/scheme part. It is the affine/scheme group that equals $W(D_5)$.) Edges of the Clebsch graph: $|E(R_4)| = \frac{16 \cdot 5}{2} = 40 = |R(D_5)| = \sum_{f,j} L_{f,j}$ (the entire flavor transport budget). So the five-slot even code is the half-spin weight structure of $\mathfrak{so}_{10}$, the classical home of one SM family with ν^c.

Theorem 3 — the Coxeter closure equation selects $g_{\mathrm{car}} = 5$

Requiring the even family to equal the action truncation through pair degree, $\dim S^+ = 2^{g-1} = \binom{g}{0} + \binom{g}{1} + \binom{g}{2}$, i.e.

$$\boxed{2^g - 2 = g(g+1) \iff g = 5} \quad [\mathrm{I}], \tag{75}$$

the unique positive-integer solution. At $g = 5$ the two sides equal 30, so *carrier exhaustion* $2^g - 2$ and the *Coxeter period* $g(g+1)$ coincide only at five slots: $2^{g_{\mathrm{car}}} - 2 = 30 = h$ and $g_{\mathrm{car}}(g_{\mathrm{car}} + 1) = 30$.

Theorem 4 — the $\mathbb{Z}_{30}$ cyclotomic Coxeter compiler

From the three atoms $(2, 3, g_{\mathrm{car}})$, with $h = 2 \cdot 3 \cdot g_{\mathrm{car}} = 30$ and $r = \varphi(h) = 8$:

$$\boxed{|R(E_8)| = rh = 240, \qquad \dim E_8 = r(h+1) = 8 \cdot 31 = 248, \qquad h + 1 = 2^{g_{\mathrm{car}}} - 1 = 31} \quad [\mathrm{I}]. \tag{76}$$

The E_8 exponents are the primitive residues modulo 30,

$$\mathrm{Exp}(E_8) = (\mathbb{Z}/30\mathbb{Z})^\times = \{1, 7, 11, 13, 17, 19, 23, 29\}, \qquad \chi_C(t) = \Phi_{30}(t) = t^8 + t^7 - t^5 - t^4 - t^3 + t + 1, \tag{77}$$

and $|(\mathbb{Z}/30)^\times| = \varphi(30) = 8 = r$. The local SM algebra and the root atlas are the two arithmetic projections of $(r, h) = (8, 30)$:

$$\Omega_{\mathrm{adm}} = 6r = 48, \quad D_{\mathrm{start}} = 2h = 60, \quad \gcd(6r, 2h) = \dim \mathfrak{g}_{\mathrm{SM}} = 12, \quad \mathrm{lcm}(6r, 2h) = rh = 240. \tag{78}$$

Theorem 5 — the root transition atlas and the positive-edge moment

The directed even-flips of C^+ are the complete graph K_{16}:

$$\boxed{\begin{gathered} |R^+(E_8)| = \binom{\dim S^+}{2} = \binom{16}{2} = 120 = \mathrm{Tr}_{S^+} X^2 = 5! = \sum_{m \in (\mathbb{Z}/30)^\times} m, \\ |R(E_8)| = 2\binom{16}{2} = 240 \end{gathered}} \quad [\mathrm{I}]. \tag{79}$$

($X = 6Y$.) So the positive E_8 roots are the unordered transitions of one family; the second hypercharge moment, the positive-root count and the sum of the E_8 exponents coincide at 120. The CP pairing $m \leftrightarrow 30 - m$ of the exponents gives four pairs, each of sum $h = 30$:

$$\boxed{\mathrm{Tr}_{S^+} X^2 = (N_{\mathrm{fam}} + N_\Phi)\, h = 4 \cdot 30 = 120 = \dim \mathfrak{so}_{16} = \underbrace{45}_{D_5} + \underbrace{15}_{A_3} + \underbrace{60}_{(10,6)}} \quad [\mathrm{I}], \tag{80}$$

so the hypercharge moment is simultaneously four CP-paired Coxeter periods *and* the adjoint half $\mathfrak{so}_{16} = (45, 1) \oplus (1, 15) \oplus (10, 6)$ of the E_8 closure.

How the root *origin* is now obtained — see Theorem 7

A naive flip-size-preserving map $R_{\text{flip}} \to \mathbb{R}^8$ cannot be E_8: the flip split $240 = 160 + 80$ ($|F| = 2, 4$) differs from the D_8 split $112 + 128$. The correct route is *not* a direct 240-metrization but the standard branching $E_8 \supset D_5 \times A_3$ (Theorem 7), which builds the root spaces from known D_5 and A_3 data. Until the A_3 lattice is derived from the four punctures (Theorem 7, [A]), E_8 remains a Coxeter atlas with a concrete, falsifiable lift.

Theorem 6 — downstream readouts from (h, r, c_3)

All discrete budgets and the seam seed are functions of $(h, r, c_3) = (30, 8, 1/(8\pi))$:

$$\dim S^+ = 2r = 16, \qquad h = 2(\dim S^+ - 1) = 30, \tag{81}$$

$$10b_1 = 5r + 1 = 41 = 1 + h\tfrac{\dim S^+}{\dim \mathfrak{g}_{\text{SM}}}, \qquad I_1^{\text{EW}} = \frac{5r+1}{6r} = \frac{41}{48}, \tag{82}$$

$$u = \frac{r}{6}c_3 + 6r\,c_3^4 = \frac{4}{3}c_3 + 48c_3^4, \qquad \delta_2 = 4!\binom{16}{2}c_3^8 = 2880\,c_3^8, \tag{83}$$

$$\xi_{\text{tree}} = \frac{6}{r} = \frac{\dim \mathfrak{g}_{\text{SM}}}{\dim S^+} = \frac{3}{4}, \qquad \frac{M_{\text{scal}}^2}{\bar{M}_{\text{Pl}}^2} = c_3^{\Omega_{\text{adm}} - 10b_1} = c_3^{r-1} = c_3^7. \tag{84}$$

Two-graph product table. With the carrier-pair graph K_5 ($|E(K_5)| = \binom{5}{2} = 10 = \mathcal{A}_\Lambda$) and the family-corner graph K_4 on μ_4 ($|E(K_4)| = \binom{4}{2} = 6$, $|R(A_3)| = 2|E(K_4)| = 12$), every large budget is a product of the two graphs:

$$\boxed{40 = 10 \cdot 4, \quad 60 = 10 \cdot 6, \quad 120 = 10 \cdot 12, \quad 240 = 10 \cdot 24} \quad \text{[I]}, \tag{85}$$

i.e. $(\sum L,\ D_{\text{start}},\ |R^+(E_8)|,\ |R(E_8)|) = 10 \cdot (|\mu_4|, |E(K_4)|, |R(A_3)|, |S_4|)$. Two occupancy identities follow,

$$\begin{gathered}\boxed{10b_1 = 1 + |\mu_4|\,|E(K_5)| = 1 + 4 \cdot 10 = 41},\\ \boxed{\Omega_{\text{adm}} = N_{\text{fam}} \dim S^+ = (N_{\text{fam}} + N_\Phi)|R(A_3)| = 3 \cdot 16 = 4 \cdot 12 = 48} \quad \text{[I]},\end{gathered} \tag{86}$$

so the Higgs singleton is the fourth (μ_4) direction that closes $N_{\text{fam}} = 3$ to **4** of A_3. **Occupancy deficit ladder.** The admissible occupancy splits cleanly,

$$\boxed{\begin{aligned}&\Omega_{\text{adm}} = \underbrace{40}_{\sum L\,(\text{flavor})} + \underbrace{1}_{\text{Higgs}} + \underbrace{7}_{\text{scalaron}} = 48, \qquad \Omega_{\text{adm}} - \sum L = 8 = h(D_5) = r(E_8),\\ &\Omega_{\text{adm}} - 10b_1 = 7 = h(D_5) - 1\end{aligned}} \quad \text{[I]}, \tag{87}$$

so the scalaron exponent $r - 1 = 7$ is exactly the post-flavor, post-Higgs occupancy remainder, $M_{\text{scal}}^2/\bar{M}_{\text{Pl}}^2 = c_3^{\Omega_{\text{adm}} - 10b_1} = c_3^7$. With $h(D_5) = 8$ and $|R^+(A_3)| = 6$ the seed itself is a D_5/A_3 ratio–product, $u = \frac{h(D_5)}{|R^+(A_3)|}c_3 + h(D_5)|R^+(A_3)|c_3^4$, and $\xi = \frac{|R^+(A_3)|}{h(D_5)}\,(1 + |R^+(A_3)|^2 c_3^3)^{-1} = \frac{3}{4}(1 + 36c_3^3)^{-1}$. The seed readouts and the two decuples are

$$\lambda_C^2 = u(1-u), \quad \beta_{\text{rad}} = 2c_3 u, \quad \sin^2\theta_{13} = e^{-5/6}u, \quad \boxed{\omega_b = \frac{\lambda_C}{\mathcal{A}_\Lambda} = \frac{\lambda_C}{10} = 0.02244}\ (\text{Planck } 0.02237), \tag{88}$$

$$\boxed{\begin{aligned}&\det M_\ell \propto u^{h/N_{\text{fam}}} = u^{10},\\ &\frac{\rho_\Lambda}{\bar{M}_{\text{Pl}}^4} = \frac{3}{4\pi^2}e^{2\delta_{\text{ph}}}\left[\frac{v_{\text{geo}}}{5\beta_{\text{rad}}^2\bar{M}_{\text{Pl}}}\right]^{h/N_{\text{fam}}}\end{aligned}}, \tag{89}$$

with lepton powers $(5,3,2)=(g_{\rm car},b,s)$: $\sum=10=\mathcal{A}_\Lambda$ (the Λ decuple), $\prod=30=h$ (the Coxeter period). The first-generation transport sums read the non-primitive $\mathbb{Z}_{30}$ load,

$$(G_1,G_2,G_3)=(22,13,5)=(h-r,\ \dim\mathfrak{g}_{\rm SM}+1,\ g_{\rm car}),\qquad h-r=30-\varphi(30)=22, \tag{90}$$

Flavor transport is a $D_5\times A_3$ object. The transport packets $L_u=(7,3,0)$, $L_d=(7,5,2)$, $L_e=(8,5,3)$ split into a D_5 budget and an A_3 geometry:

$$\boxed{\textstyle\sum_{f,j}L_{f,j}=40=|R(D_5)|\ \text{(Clebsch edges)},\qquad L_{f,j}\bmod 6\ \text{lives on}\ C_6=Z_6=|R^+(A_3)|}\qquad \text{[I]}, \tag{91}$$

the residues being the D_6-orbit of the perfect set $\{0,1,3\}\subset C_6$ (cyclic distances $\{1,2,3\}$): $u\equiv\{0,1,3\}$, $d\equiv 2-\{0,1,3\}$, $e\equiv 2+\{0,1,3\}$, with the universal first-generation $+6$ winding. So the carrier D_5 sets the total budget and the family A_3 (its 6-element positive-root hexagon) sets the per-entry distances. The scalaron mass is a pure seam power [I]; the amplitude is a *conditional* readout, kept [P]/[A] until the scalaron variable is formally identified with the Starobinsky M parameter. That identification is now *heat-kernel grounded* (G2): the spectral-action Seeley–DeWitt coefficient $a_4|_{R^2}\sim R^2/72$ delivers exactly the Starobinsky R^2 term, with $M^2/\bar M_{\rm Pl}^2=6(4\pi)^2/f_0$ and the TFPT closure c_3^7 fixing the cutoff moment f_0 ([`verification/v36_spectral_action_g2.py`]):

$$\underbrace{\frac{M^2_{\rm scal}}{\bar M^2_{\rm Pl}}=c_3^{\,r-1}=c_3^7,}_{\text{[I]}}\qquad \underbrace{A_s=\frac{N_\star^2}{24\pi^2}\,\frac{M^2_{\rm scal}}{\bar M^2_{\rm Pl}}=\frac{N_\star^2c_3^7}{24\pi^2}=2.17\times10^{-9}\,.}_{\text{[P]/[A] (if } M_{\rm scal}=M_{\rm Star})} \tag{92}$$

Theorem 7 — the $D_5\times A_3$ lift of the E_8 atlas

Since $A_3=\mathfrak{su}_4\cong\mathfrak{so}_6=D_3$, one has the standard chain $E_8\supset\mathfrak{so}_{16}\supset\mathfrak{so}_{10}\oplus\mathfrak{so}_6=D_5\times A_3$, with the exact branching (verified dimensions and root/weight counts):

$$\begin{gathered}\boxed{\mathfrak{e}_8=(45,1)\oplus(1,15)\oplus(10,6)\oplus(16,4)\oplus(\overline{16},\overline{4})},\\ \underbrace{45+15+60+64+64}_{=248}=\dim E_8,\qquad \underbrace{40+12+60+64+64}_{=240}=|R(E_8)|.\end{gathered} \tag{93}$$

The adjoint **120** of $\mathfrak{so}_{16}$ is $(45,1)\oplus(1,15)\oplus(10,6)$ and its spinor **128** is $(16,4)\oplus(\overline{16},\overline{4})$. The TFPT dictionary makes each summand a known object:

piece	value	TFPT reading
$(45,1)$	$45=40+5$	D_5 carrier adjoint: Clebsch edges $40=\lvert R(D_5)\rvert$ plus carrier rank $g_{\rm car}=5$
$(1,15)$	$15=12+3$	A_3 family+Higgs adjoint: $\dim\mathfrak{g}_{\rm SM}=12$ plus family rank $N_{\rm fam}=3$
$(10,6)$	$10\cdot 6$	Λ decuple $\mathcal{A}_\Lambda=\binom{5}{2}=10$ times Z_6 transport $\binom{4}{2}=6$
$(16,4)\oplus(\overline{16},\overline{4})$	$2\cdot 64$	matter half-spinor S^+ times the four corners μ_4, plus conjugate sheet

Proof route (replaces the old blind 240-metrization). *Stage 1 (D_5, [I]):* C^+ is the D_5 half-spin orbit, $R_4=$ Clebsch, $\mathrm{Aut}=W(D_5)$, $|E(R_4)|=40=|R(D_5)|$ — Theorem 2. *Stage 2 (A_3, [I]):* Paper 2 fixes $X_f^\circ=\mathbb{P}^1\setminus\mu_4$ with $H_1(X_f^\circ,\mathbb{Z})=\mathbb{Z}^3$ and a rigid $D_4\subset PGL_2$ generated by $z\mapsto iz$ and $z\mapsto 1/z$. The logarithmic forms $\omega_i=dz/(z-p_i)$ regular at infinity span H^1 by their differences, dual under the residue pairing to the puncture loops γ_i with $\mathrm{Res}(\omega_i,\gamma_j)=\delta_{ij}$; hence the canonical (residue) metric on the puncture classes is I_4, and the twelve puncture-difference cycles $\gamma_i-\gamma_j$ are exactly the A_3 roots:

$$\boxed{\begin{array}{l}\{\gamma_i-\gamma_j\}_{i\neq j}:\ \#=12=|R(A_3)|=\dim\mathfrak{g}_{\rm SM},\quad \mathrm{rank}=3=N_{\rm fam},\\ \mathrm{Gram}=\mathrm{Cartan}(A_3),\ \det=4\end{array}}\qquad\text{[I]}, \tag{94}$$

with $\dim A_3 = 15 = \dim S^+ - 1$, and the A_3 positive roots equal the exponent sum $1{+}2{+}3 = 6 = Z_6 = |E(K_4)|$. *The corner rotation realises the A_3 Coxeter element.* The Paper-2 generator $z \mapsto iz$ permutes μ_4 as the 4-cycle $(p_1p_2p_3p_4) \in S_4 = W(A_3)$ and acts on $H^1 = \mathbb{C}^3$ as an isometry of the Cartan form with

$$\boxed{\operatorname{ord} = 4 = h(A_3) = |\mu_4|, \quad \operatorname{spec} = \{i, -1, -i\}, \quad \chi(t) = \tfrac{t^4-1}{t-1} = \Phi_4(t)\Phi_2(t)} \qquad [\mathbf{I}], \tag{95}$$

i.e. the eigenvalues are the A_3 exponents $\{1, 2, 3\}$ as fourth roots of unity — the exact analogue of the E_8 Coxeter element reading Φ_{30}. The full $D_4 = \langle z \mapsto iz, z \mapsto 1/z\rangle$ embeds in $W(A_3) = S_4$ (order 8, index 3) by isometries. *Stage 3 (standard, [I]):* assemble E_8 from the $D_5 \times A_3$ branching — no 240-map required.

The μ_4 glue theorem (why the four corners appear everywhere). The two factors have equal discriminant groups, and E_8 is unimodular:

$$\boxed{\operatorname{disc}(D_5) = \operatorname{disc}(A_3) = \mathbb{Z}_4, \qquad [\,E_8 : D_5 \oplus A_3\,] = \sqrt{\tfrac{\det D_5 \cdot \det A_3}{\det E_8}} = \sqrt{\tfrac{4\cdot 4}{1}} = 4 = |\mu_4|} \qquad [\mathbf{I}]. \tag{96}$$

So $E_8 = (D_5 \oplus A_3) + \mu_4$*-glue*: the four family corners $\mu_4 = \mathbb{Z}_4$ are precisely the glue code that fuses the carrier and the family into the unimodular E_8 lattice. The glue pairs the D_5 spinor class ($= S^+$, the family) with the A_3 class — exactly the $(16, 4) \oplus (\overline{16}, \overline{4})$ spinor block. Hence μ_4 plays three roles at once: family corners, A_3 origin, and E_8 glue.

Discriminant-norm glue lemma (and the TFPT $5/4, 3/4$ identification). The two glue generators carry the discriminant quadratic form $q \in \mathbb{Q}/2\mathbb{Z}$:

$$\boxed{\begin{array}{ll} q(D_5) = \tfrac{n}{4}\big|_{n=5} = \tfrac{5}{4} \text{ (spinor class)}, & q(A_3) = \tfrac{n}{n+1}\big|_{n=3} = \tfrac{3}{4} \text{ (corner class)}, \\ q(D_5) + q(A_3) = 2 \equiv 0 \pmod 2, & q(D_5) - q(A_3) = \tfrac{1}{2} \end{array}} \qquad [\mathbf{I}]. \tag{97}$$

The diagonal $\langle (s, g)\rangle \cong \mathbb{Z}_4$ is isotropic ($k^2 \cdot 2 \equiv 0$ for all k), so the glue closes to an *even unimodular* lattice — this is E_8.

The two glue norms are just $(g_{\rm car}, N_{\rm fam})$ over the glue index. Because $\operatorname{rank} D_5 = g_{\rm car} = 5$ (so $q(D_5) = \frac{n}{4}|_{n=5} = g_{\rm car}/|\mu_4|$) and $\operatorname{rank} A_3 = N_{\rm fam} = 3$ (so $q(A_3) = \frac{n}{n+1}|_{n=3} = N_{\rm fam}/|\mu_4|$, with $n{+}1 = 4 = |\mu_4|$),

$$\boxed{q(D_5) = \frac{g_{\rm car}}{|\mu_4|} = \tfrac{5}{4}, \qquad q(A_3) = \frac{N_{\rm fam}}{|\mu_4|} = \tfrac{3}{4}}$$

and *all four arithmetic operations* on the pair are then forced compiler atoms:

$$\underbrace{q(D_5){+}q(A_3) = \tfrac{g_{\rm car}+N_{\rm fam}}{|\mu_4|} = 2}_{E_8 \text{ root norm}}, \qquad \underbrace{q(D_5){-}q(A_3) = \tfrac{|\mathbb{Z}_2|}{|\mu_4|} = \tfrac{1}{2}}_{\delta \text{ (lepton transport)}},$$

$$\underbrace{q(D_5)q(A_3) = \tfrac{g_{\rm car}N_{\rm fam}}{|\mu_4|^2} = \tfrac{15}{16}}_{\dim \mathfrak{su}(4)/\dim S^+}, \qquad \underbrace{\tfrac{q(D_5)}{q(A_3)} = \tfrac{g_{\rm car}}{N_{\rm fam}} = \tfrac{5}{3}} .$$

So the distinguished lepton transport value $\delta = \frac{1}{2} = (g_{\rm car} - N_{\rm fam})/|\mu_4| = |\mathbb{Z}_2|/|\mu_4|$ is the carrier-minus-family count over the glue index — a fully atomic origin, not a chosen rational (it also equals the harmonic-frame holonomy diagonal modulus, [`verification/v41_leg_assignment.py`]); and (sum, ratio) invert uniquely back to $(\frac{5}{4}, \frac{3}{4})$. [`verification/v51_boundary_half_step.py`] The sum $\frac{5}{4}{+}\frac{3}{4} = 2$ is literally the E_8 spinor-root norm $\frac{1}{2}(\pm)^8$ split across $D_5 \times D_3$: $5 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = 2$. Strikingly, both norms are pre-existing TFPT constants:

$$\boxed{\frac{\delta_2}{\delta_{\rm top}^2} = \frac{2880\, c_3^8}{(48\, c_3^4)^2} = \frac{5}{4} = q(D_5), \qquad \xi_{\rm tree} = \frac{\dim \mathfrak{g}_{\rm SM}}{\dim S^+} = \frac{12}{16} = \frac{3}{4} = q(A_3)} \qquad [\mathbf{I}], \tag{98}$$

so $\delta_2/\delta_{\rm top}^2 + \xi_{\rm tree} = 2$: the two-loop/defect ratio is the D_5 spinor norm and the gravitational tree normalization is the A_3 corner norm, and their sum is the E_8 root norm.

The whole integer skeleton from one pair $(g_{\rm car}, N_{\rm fam}) = (5,3)$ [I] ([`verification/v53_compiler_core.py`])

Every load-bearing integer of the compiler flows from the single boundary pair $(g_{\rm car}, N_{\rm fam}) = (5,3)$ by sum, difference and squares:

$$\mathrm{rank}\,E_8 = g_{\rm car}+N_{\rm fam} = 8, \qquad |\mathbb{Z}_2| = g_{\rm car}-N_{\rm fam} = 2, \qquad |\mu_4| = \tfrac{g_{\rm car}+N_{\rm fam}}{2} = 4,$$

and $(N_{\rm fam}, |\mu_4|, g_{\rm car}) = (3,4,5)$ is *the* Pythagorean triple. This is not decorative: the grand-mass-volume exponent $\Delta_Y = g_{\rm car}^2 = 25$ ($\det M_{\rm SM} \sim (\varphi_0^{\rm ret})^{25}$, §Paper 2) splits over the sectors as a *difference of squares*,

$$\boxed{\Delta_Y = g_{\rm car}^2 = \underbrace{N_{\rm fam}^2}_{\text{down}=9} + \underbrace{(g_{\rm car}-N_{\rm fam})(g_{\rm car}+N_{\rm fam})}_{=|\mathbb{Z}_2|\cdot\mathrm{rank}\,E_8} = 9+16 = \dim S^+}$$

i.e. the down sector carries $N_{\rm fam}^2$ and up+lepton carries $|\mathbb{Z}_2|\cdot\mathrm{rank}\,E_8 = \dim S^+$ (the K-row sums $(6,9,10)$). **The anchor is the Dirac square root of this skeleton:** $a = (1,1,2)$ is the *unique* 3-multiset whose elementary symmetric functions are $(|\mu_4|, g_{\rm car}, |\mathbb{Z}_2|) = (4,5,2)$, equivalently its characteristic polynomial is $\chi_a(t) = (t-1)^2(t-2) = t^3 - |\mu_4|\,t^2 + g_{\rm car}\,t - |\mathbb{Z}_2|$ — the compiler atoms *are* its char-poly coefficients. So the theory's only transcendental is π (through $c_3 = \frac{1}{8\pi}$, $8 = \mathrm{rank}\,E_8$, and the seed $\frac{1}{6\pi}$, $6 = |R^+(A_3)|$); everything else is this $(5,3)$-generated integer skeleton.

Boundary Carrier Selection Theorem (Theorem A) /[P]

E_8 is not assumed: it is *selected* as the unique even-unimodular audit hull of the admissible boundary-carrier data. **Boundary side:** $\mu_4 = \{1, i, -1, -i\}$ gives $X_f = \mathbb{P}^1 \setminus \mu_4$ with $H_1 = \mathbb{Z}^3$, hence A_3 and $N_{\rm fam} = 3$. **Carrier side:** a minimal D-type carrier with a 16-dim half-spinor forces D_5 (only $n = 5$ gives $2^{n-1} = 16$; $S^+ = \Lambda^{\rm even}\mathbb{C}^5 = 1{+}10{+}5$). **Glue:** $\mathrm{disc}(D_5) = \mathrm{disc}(A_3) = \mathbb{Z}_4$ (isomorphic, cyclic, anti-isometric), $q(D_5) + q(A_3) = 2$ (the E_8 root norm), glue index $4 = |\mu_4|$, and E_8 is the unique even unimodular rank-8 lattice. **Exclusion:** the four conditions

$$\text{(i)}\ \dim S^+{=}16, \quad \text{(ii)}\ N_{\rm fam}{=}3, \quad \text{(iii) cyclic } \mathbb{Z}_4 \text{ glue}, \quad \text{(iv)}\ q_D{+}q_A{=}2$$

are met *only* by $D_5 \oplus A_3$; D_8 (non-cyclic $\mathbb{Z}_2{\times}\mathbb{Z}_2$, no family geometry), $E_7{\oplus}A_1$ (1 family, no 16-spinor), $E_6{\oplus}A_2$ (2 families, $\mathbb{Z}_3$), $A_4{\oplus}A_4$ (no D-spinor, $\mathbb{Z}_5$) and A_8 (single factor) each fail at least one. **Typing:** the lattice/classification facts are [L]; the physical selection (reflection-positive seam $c_3 = 1/8\pi$ + minimal chiral carrier ⇒ exactly these inputs) is the [P]/[A] selection axiom (P1 analytic interface + P2 carrier interface), not a zero-input derivation. [`verification/v47_selection_theorem.py`]

Group-level closure (μ_4 as the common centre). The lattice glue lifts to a clean group statement, because both factors share the *same* centre and μ_4 *is* the diagonal:

$$\boxed{Z(\mathrm{Spin}(10)) = \mathbb{Z}_4, \quad Z(SU(4)) = \mathbb{Z}_4, \quad |\mu_4| = 4, \quad G_{\rm closure} = \frac{\mathrm{Spin}(10)\times SU(4)}{\Delta\mathbb{Z}_4} \subset E_8} \quad [\mathrm{I}], \tag{99}$$

with rank $= 5+3 = 8 = \mathrm{rank}\,E_8$. Every adjoint block is diagonal-$\mathbb{Z}_4$ neutral (generator $\omega = i$; charges mod 4): $(45,1), (1,15) : 0$; $(10,6) : 2{+}2{=}0$; $(16,4) : 1{+}3{=}0$; $(\overline{16},\overline{4}) : 3{+}1{=}0$, so $45{+}15{+}60{+}64{+}64 = 248$ are genuine reps of the quotient. This promotes μ_4 from a lattice glue index to the *common centre quotient* — the physically cleanest language for the gluing: the D_5 half-spinor 16 pairs with the $SU(4)$ fundamental 4 so the spinor charges ±1 cancel the fundamental charges ∓ 1.

Root-length completion. Every E_8 block reaches norm 2:

block	norm split	reading
$(45,1),(1,15)$	$2,\ 2$	$D_5,\ A_3$ root adjoints
$(10,6)$	$1+1=2$	$SO(10)$ vector $\oplus$ $SO(6)$ vector
$(16,4)\oplus(\overline{16},\overline{4})$	$\frac{5}{4}+\frac{3}{4}=2$	D_5 spinor norm + A_3 corner norm

The old question "can the 240 flip atlas be metrized to E_8?" is now the sharp question "are the TFPT D_5 and A_3 metrics in standard normalization?" — and the $5/4, 3/4$ match says yes.

Explicit construction (the lift is now built, not counted). Embedding $D_5 \oplus A_3 = D_5 \oplus D_3 \subset D_8$ and adjoining the even spinor coset (the μ_4 glue), the 240 roots are produced explicitly and verified:

$$\boxed{\{\pm e_i \pm e_j\}_{i<j}^{8}\ (112)\ \cup\ \{\tfrac{1}{2}(\pm)^8 :\ \text{even}\}\ (128)\ =\ 240 \text{ roots, all norm } 2,\ \text{Cartan}\ \det = 1} \qquad [\mathbf{L}]. \tag{100}$$

Splitting each root into its D_5 (first five) and A_3 (last three) parts gives exactly the branching blocks — $(2,0)\!:\!40$, $(0,2)\!:\!12$, $(1,1)\!:\!60$, and $(\frac{5}{4},\frac{3}{4})\!:\!128$ — and the 128 spinor roots realise $\frac{5}{4}+\frac{3}{4}=2$ on the nose. All 240 vectors are integer combinations of one set of 8 simple roots, so the assembled lattice *is* E_8. The blind 240-metrization is therefore eliminated: E_8 is explicitly $(D_5 \oplus A_3) + \mu_4$-glue.

Remark (the metric, computed)**.** This reframes E_8 from origin to closure: E_8 is the closure algebra of the D_5 carrier and the A_3 family geometry, glued by μ_4. The root system, Cartan form, $W(A_3) = S_4$ and the corner-rotation $= A_3$ Coxeter element are exact (residue pairing + explicit Paper-2 symmetry). The transcendental geometric metric $G_{ij} = -\log|p_i - p_j|$ on μ_4 has been computed: it is only D_4-symmetric, with a regularisation-independent anisotropy $-\log 2$ on the root space, hence *not* proportional to Cartan(A_3). It need not be: the E_8-relevant A_3 is the canonical *residue* pairing, and the geometric metric is a D_4-equivariant deformation that preserves the root system, Weyl group and Coxeter element. So the lattice data is exact [I]; the analytic metric is a *characterised* deformation, not an open gap.

Uniform Coxeter law. All three factors obey $|R| = h \cdot \text{rank}$, and the two TFPT Coxeter numbers are themselves carrier data:

$$\boxed{\begin{aligned} &h(D_5) = 8 = r(E_8) = \varphi(30), \quad h(A_3) = 4 = |\mu_4|, \\ &h(D_5)\,h(A_3) = 2^{g_{\text{car}}} = 32, \quad 2^{g_{\text{car}}} - 2 = 30 = h(E_8) \end{aligned}} \qquad [\mathbf{I}], \tag{101}$$

so the Coxeter closure of Theorem 3 reappears as $h(E_8) = h(D_5)h(A_3) - 2$, and $h(D_5) + h(A_3) = 12 = \dim \mathfrak{g}_{\text{SM}} = |R(A_3)|$.

Theorem 8 — the $(1,1,2)$ anchor microcode and the φ_0-ladder

The parabolic flavor bundle $E_\bullet = \mathcal{O}(-2) \oplus \mathcal{O}(-1)^2$ on $\mathbb{P}^1 \setminus \mu_4$ has splitting type $a = (1,1,2)$, whose power sums $p_n(a) = 2 + 2^n$ generate the whole compiler budget:

The anchor power compiler

$$\begin{gathered} (p_1, \ldots, p_5) = (4, 6, 10, 18, 34) = (|\mu_4|,\ |R^+(A_3)|,\ \mathcal{A}_\Lambda,\ N_{\text{fam}}|R^+(A_3)|,\ \mathbb{Z}_6 \text{ lift}), \\ q(A_3) = \frac{p_2}{2p_1} = \frac{3}{4}, \quad q(D_5) = \frac{p_3}{2p_1} = \frac{5}{4}, \quad \textstyle\sum L = p_1 p_3 = 40, \\ \Omega_{\text{adm}} = 2p_1p_2 = 48, \qquad D_{\text{start}} = p_2p_3 = 60, \qquad |R(E_8)| = 4p_2p_3 = 240. \end{gathered}$$

The elementary symmetric data $\chi_a(t) = (t-1)^2(t-2)$ give $(e_1, e_2, e_3) = (4, 5, 2) = (|\mu_4|, g_{\text{car}}, |\mathbb{Z}_2|)$, so the anchor reconstructs the carrier rank, and $10b_1 = e_1^2 + e_2^2 = 4^2 + 5^2 = 41$.

The seed itself is the two-term anchor form $\varphi_0^{\mathrm{ret}} = \frac{2p_1}{p_2} c_3 + 2p_1p_2\, c_3^4 = \frac{4}{3} c_3 + 48c_3^4$.

The φ_0-ladder. With $\lambda_Y = \sqrt{\varphi_0^{\mathrm{ret}}(1-\varphi_0^{\mathrm{ret}})}$, every fermion mass is the single closed form $\hat{m}_{f,j} = \frac{v_{\mathrm{geo}}}{\sqrt{2}} \lambda_Y^{L_{f,j}} \Lambda_{f,j}$ with $\lambda_Y^L \Lambda = \pi\, c\,(\varphi_0^{\mathrm{ret}})^k$, so all nine holonomies are resolvent outputs and the masses form the φ_0^{ret}-order ladder $t(0) > \{b, c, \tau\}(2) > \{s, \mu\}(3) > \{u, d\}(4) > e(5)$.
Secondary E_8 atlas. Beyond the main $D_5 \times A_3$ split, the residue matrix appears in the $E_6 \times A_2$ branching, $248 = \|R\|_F^2 + \det R + 2(\mathbf{1}^\top Ra)N_{\mathrm{fam}} = 78+8+2{\cdot}27{\cdot}3$, and the scalaron exponent in $E_7 \times A_1$, $112 = 7{\cdot}16$. The seven E_8 slices are charted in the companion note `e8_secondary_branching_atlas` as a falsifiable audit raster.

Master statement and status

One line

$$\boxed{\text{Five carrier slots give } D_5,\ \text{four family corners give } A_3,\ \text{and } D_5 \times A_3 \text{ gives the } E_8 \text{ atlas.}}$$

Equivalently: $\mathcal{A}_\Lambda = \frac{h}{N_{\mathrm{fam}}} = 10 = \binom{5}{2}$, $h = 30 = 2 \cdot 3 \cdot 5$, $(5, 3, 2) : \ \sum = 10,\ \prod = 30$; Λ, $\det M_\ell$, the E_8 Coxeter structure and the action ladder read the same five-slot pair-decuple. The Standard model, flavor, Λ, inflation and E_8 are projections of the even carrier code C^+ ($= D_5$ half-spinor) on five slots together with the four-point family A_3.

Statement	Status	Note
Even code C^+, $\dim S^+ = 16$, dual-root Y	[I]	Theorem 1
Exterior–Action duality $(1, 10, 5) \leftrightarrow (1, 5, 10)$	[I]	scheme eigenvalues verified
R_4 = Clebsch $(16, 5, 0, 2)$, $R_2 = (16, 10, 6, 6)$; $\mathrm{Aut}_{\mathrm{aff}} = W(D_5) = 1920$	[I]	carrier = D_5 half-spinor
$2^g - 2 = g(g+1) \Leftrightarrow g = 5$	[I]	Coxeter closure (unique)
$h{=}30{=}2{\cdot}3{\cdot}5$, $r{=}\varphi(30){=}8$, $\chi_C{=}\Phi_{30}$	[I]/[P]	cyclotomic compiler
$\lvert R^+\rvert = \binom{16}{2} = \mathrm{Tr}\, X^2 = 4h = 120$	[I]	positive-edge moment
$40, 60, 120, 240 = 10 \cdot (4, 6, 12, 24)$; $41{=}1{+}4{\cdot}10$	[I]	$K_5 \times K_4$ product table
$10b_1{=}5r{+}1$, u totient, $M_{\mathrm{scal}}^2{=}c_3^{r-1}$, $\omega_b{=}\lambda_C/10$	[I]/[P]	downstream readouts
$A_s = N_\star^2 c_3^7 / 24\pi^2$	[P]/[A]	if $M_{\mathrm{scal}} = M_{\mathrm{Star}}$
$\mathfrak{e}_8 = (45, 1) \oplus (1, 15) \oplus (10, 6) \oplus (16, 4) \oplus (\overline{16}, \overline{4})$	[I]/[P]	$D_5 \times A_3$ lift
240 roots built from $D_5{\oplus}A_3{+}\mu_4$ glue, $\det 1$	[L]	explicit E_8 construction
flavor $\sum L{=}40{=}\lvert R(D_5)\rvert$, entries on $C_6{=}\lvert R^+(A_3)\rvert$	[I]	transport = $D_5{\times}A_3$
A_3 from $\mathbb{P}^1 \setminus \mu_4$: residue pairing = Cartan, $\det 4$	[I]	Stage 2 (Theorem 7)
corner $z \mapsto iz$ = A_3 Coxeter el.: $\{i, -1, -i\}$, $\Phi_4\Phi_2$	[I]	order $4 = h(A_3) = \lvert\mu_4\rvert$
$\mathrm{disc}(D_5) = \mathrm{disc}(A_3) = \mathbb{Z}_4$, glue $= \mu_4$	[I]	$E_8 = (D_5{\oplus}A_3){+}\mu_4$
$q(D_5){=}\frac{5}{4}{=}\delta_2/\delta_{\mathrm{top}}^2$, $q(A_3){=}\frac{3}{4}{=}\xi_{\mathrm{tree}}$, sum 2	[I]	discriminant-norm glue
$120{=}\dim \mathfrak{so}_{16}$; $\Omega_{\mathrm{adm}}{=}40{+}1{+}7$	[I]	adjoint half; deficit ladder
$h(D_5){=}8{=}r(E_8)$, $h(A_3){=}4{=}\lvert\mu_4\rvert$, $\lvert R\rvert{=}h \cdot r$	[I]	uniform Coxeter law
$\mathrm{Exp}(E_8)$ as 41/48 budget deficits	[A]	exponent ledger (fingerprint)
geometric metric D_4-deform (anisotropy $-\log 2$), residue $= A_3$	[I]	metric computed/characterised

Scope. This note compresses; it does not re-derive the upstream boundary kernel (Paper 1) or the carrier rigidity theorem (Paper 2), and it makes no claim of a final theory. The arithmetic and the D_5/Clebsch/$W(D_5)$, A_3/residue-Cartan/Coxeter, and μ_4-glue identifications are exact, and the E_8 lattice is now built explicitly from $D_5 \oplus A_3$ plus the μ_4 glue (240 roots, det 1). The only residual [P] is the analytic identification of Paper 2's L^2 cusp Hodge metric with the canonical residue metric up to scale, which does not affect the explicit root/lattice/Weyl construction. Honest non-fits (Koide $0.6645 \neq 2/3$, η_B, m_p/m_e) stay outside the compiler.

Appendix: machine-checkable $E_8 = (D_5 \oplus A_3) + \mu_4$ construction

The lift is reproducible by a short script (all entries below verified exactly).

Cartan matrices and discriminant groups.

$$C(D_5) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}, \quad C(A_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}; \quad \det C(D_5) = \det C(A_3) = 4.$$

Smith normal forms give $\mathrm{disc}(D_5) = \mathrm{disc}(A_3) = \mathbb{Z}_4$; the glue index is $[\,E_8 : D_5 \oplus A_3\,] = \sqrt{4 \cdot 4/1} = 4 = |\mu_4|$, with discriminant-form norms $q(D_5) = \frac{5}{4}$, $q(A_3) = \frac{3}{4}$, $q(D_5){+}q(A_3) = 2 \equiv 0 \pmod 2$ (even-glue). Writing $\mu := |\mu_4| = 4$ the norms are $q(D_5) = \frac{\mu+1}{\mu}$, $q(A_3) = \frac{\mu-1}{\mu}$, so $q(D_5)q(A_3) = 1 - \frac{1}{\mu^2} = \frac{15}{16}$ and

$$\boxed{\dim S^+(1 - q(D_5)q(A_3)) = 16 \cdot \tfrac{1}{16} = 1} \qquad [\mathbf{I}], \tag{102}$$

i.e. the glue product fills all $15 = \dim S^+{-}1$ non-trivial half-spin states and the defect is exactly the singleton (the ν^c/Higgs closure). And E_8 has a tidy μ_4-cardinal form, $\dim E_8 = \binom{|\mu_4|^2}{2} + 2|\mu_4|^3 = \binom{16}{2} + 2 \cdot 64 = 120 + 128 = 248$ [P].

Explicit roots. Split $\mathbb{R}^8 = \mathbb{R}^5_{(D_5)} \oplus \mathbb{R}^3_{(A_3)}$. The 240 roots are

$$\{\pm e_i \pm e_j : 1 \leq i < j \leq 8\}\ (112)\ \cup\ \{\tfrac{1}{2}(\pm 1)^8 :\ \text{even}\ \#\ \text{of}\ -\}\ (128),$$

all of norm 2. Splitting each by $(\|\cdot\|^2_{D_5}, \|\cdot\|^2_{A_3})$ gives the branching blocks

$$(2,0)\colon 40\ [D_5], \quad (0,2)\colon 12\ [A_3], \quad (1,1)\colon 60\ [(10,6)], \quad (\tfrac{5}{4},\tfrac{3}{4})\colon 128\ [(16,4) \oplus (\overline{16},\overline{4})],$$

so $\mathfrak{e}_8 = (45,1) \oplus (1,15) \oplus (10,6) \oplus (16,4) \oplus (\overline{16},\overline{4})$, $248 = 45{+}15{+}60{+}64{+}64$, $240 = 40{+}12{+}60{+}64{+}64$.

Simple-root Gram (unimodular check). With the Bourbaki simple roots $\alpha_1 = \frac{1}{2}(e_1{-}e_2{-}\ldots{-}e_7{+}e_8)$, $\alpha_2 = e_1{+}e_2$, and $\alpha_k = e_{k-1}{-}e_{k-2}$ for $k = 3, \ldots, 8$ (so $\alpha_3 = e_2{-}e_1$, $\alpha_4 = e_3{-}e_2$, $\ldots$, $\alpha_8 = e_7{-}e_6$), the Gram matrix $G_{ij} = \langle \alpha_i, \alpha_j \rangle$ is the E_8 Cartan matrix with $\det G = 1$ and all $G_{ii} = 2$; every one of the 240 roots is an integer combination of the α_i. Hence the assembled lattice is E_8, not merely a 240-count. [F] (The shifted index $e_{k-2}{-}e_{k-3}$ would duplicate α_3 and make G singular — use $e_{k-1}{-}e_{k-2}$.)

Part III

Appendix: ablation of the EM fixed point

The single sharpest reviewer test is: *was the closure* $F_{U(1)}(\alpha) = 0$ *engineered so that* α *comes out?* The honest answer is an ablation — vary each input and see which actually move the root. Three one-parameter scans (all others held at their compiler values) separate the *load-bearing* integers from the *fine* last-digit selector.

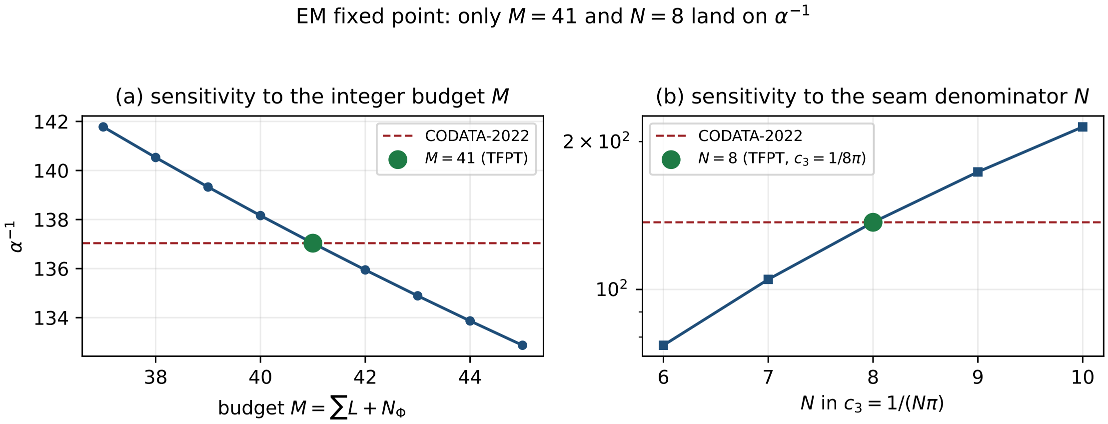


Figure 7: EM-closure ablation (data plot, `verification/make_figures.py`). **(a)** $\alpha_\star^{-1}$ vs the integer budget M: only $M = 41$ lands on CODATA-2022. **(b)** $\alpha_\star^{-1}$ vs the seam denominator N in $c_3 = 1/(N\pi)$: only $N = 8$ lands. Both integers are fixed elsewhere (flavor budget; rank/$h(D_5)$), so the equation is not freely tunable to 137.036.

(A1) The transport budget $M = \sum L + N_\Phi$ (exponent held at $-\frac{5}{4}$):

M	38	39	40	**41**	42	43	44
$\alpha_\star^{-1}$	140.530	139.326	138.162	**137.0359992**	135.946	134.890	133.866

Each unit of M shifts $\alpha_\star^{-1}$ by ≈ 1.09, so *only* $M = 41$ lands on 137.036. This is a hard lock — and $41 = 10b_1 = \sum L + N_\Phi = 40 + 1$ is fixed by the compiler (flavor budget 40 + one Higgs). **Load-bearing.**

(A2) The seam “8” in $c_3 = 1/(N\pi)$ (here $\delta_{\text{top}} = 48c_3^4$ tracks c_3):

N	6	7	**8**	9	10
$\alpha_\star^{-1}$	76.97	104.85	**137.036**	173.53	214.34

Only $N = 8$ — i.e. $c_3 = 1/(8\pi) = 1/(\text{rank}\, E_8 \cdot \pi)$ — reproduces 137.036. The “8” is the same one fixed by the bootstrap ($h(D_5) = \text{rank}\, E_8 = \varphi(30)$) and the Gauss–Bonnet hardening. **Load-bearing.**

(A3) The carrier discriminant exponent (at $M = 41$, $c_3 = 1/(8\pi)$):

exponent	$-\frac{1}{4}$	$-\frac{3}{4}$	$\mathbf{-\frac{5}{4}}$	$-\frac{7}{4}$	$-\frac{9}{4}$
$\alpha_\star^{-1}$	137.035995	137.035997	**137.0359992**	137.036001	137.036003

The exponent moves only the *seventh* digit; all candidates agree to six. Here $-\frac{5}{4} = q(D_5)$ is selected, but this is a *fine* selector, not a hard lock. **Fine (last-digit).**

Ablation verdict — honest

The EM fixed point is a **hard test of two integers** — the budget $M = 41$ and the seam denominator $8 = \operatorname{rank} E_8$, each decisive (a single-unit change misses by ~ 1 in $\alpha_\star^{-1}$) — and a **fine test of one norm**, the carrier discriminant exponent $-\frac{5}{4} = q(D_5)$ (seventh digit). The equation is therefore *not* freely tunable to 137.036: the two integers that hit it are exactly the compiler/bootstrap numbers, fixed elsewhere. What it does *not* do is pin the carrier norm strongly — that is honest, and it is the right level of claim. [N]

Appendix: computational verification

The load-bearing identities of this document (marked [I], [L], [N]) are re-derived from $\{c_3, g_{\mathrm{car}}\}$ alone and machine-checked by the suite in the repository folder `verification/`; the inline tags `[verification/...]` point to the exact script. Relevant here:

- `v1_e8_glue.py` — the μ_4 glue ($q(D_5)+q(A_3)=2$, $\operatorname{disc}=\mathbb{Z}_4$, $240, 248$).
- `v2_carrier_pascal.py` — $g_{\mathrm{car}}=5$, $16=1+5+10$, $N_{\mathrm{fam}}, \Omega_{\mathrm{adm}}, b_1$.
- `v3_em_alpha.py` — the EM fixed point $\alpha_\star^{-1}=137.0359992168$ (existence/uniqueness, ablation).
- `v4_flavor_matrix.py` — R with $\det=8$, minors $\{2,3,5\}$, $\sum L=40$.
- `v7_gravity_cosmo.py` — scalaron c_3^7, Λ/A_s readouts.

Run `python verification/run_all.py` (needs `mpmath, numpy, sympy`); it exits 0 iff all checks pass. Full map: `verification/README.md`.